On the Network Topology Dependent Solution Count of the Algebraic Load Flow Equations

Tianran Chen[*] [†]and Dhagash Mehta[‡] [§]

Abstract—**Active research activity in power systems areas has focused on developing computational methods to solve load flow equations where a key question is the maximum number of solutions. Though several upper bounds exist, recent studies have hinted that much sharper upper bounds that depend on the topology of underlying power networks may exist. This paper provides a significant refinement of these observations. We also develop a geometric construction called adjacency polytope which accurately captures the topology of a power network and is immensely useful in the computation of the solution bound. Finally we highlight the significant implications of the development of such solution bounds in numerically solving load flow equations.**

I. Introduction

Engineers are regularly required to perform power flow computations for designing, operating, and controlling power systems [1]. In this, a key mathematical problem is to solve a system of multivariate nonlinear equations known as the *load flow equations*. In general, load flow equations may have more than one solutions [2]. There are quite a few existing methods for finding one or many solutions [3]–[29] (see [30] for a recent review). Out of the few methods that guarantee to find *all* load flow solutions, i.e., the interval based approach [19], Gröbner bases technique [20]–[23] and the numerical polynomial homotopy continuation (NPHC) method [24]–[29], the NPHC method appears most promising in scalability with increasing system sizes in that it has already found all load flow solutions of up to IEEE 14-bus systems [26] (and 18 oscillators case for the Kuramoto model [29]) and is inherently parallel: formulating load flow equations as systems of polynomial equations, the NPHC method, rooted from complex algebraic geometry, finds all isolated complex solutions which obviously include all isolated real solutions. NPHC solves the algebraic versions of load flow equations by constructing smooth paths connecting each of its complex solutions to a corresponding prescribed starting point. Then numerical continuation methods can be applied to trace the paths from the known starting points to reach all the complex solutions. In this context the a priory knowledge of an upper bound on the number of solutions is crucially important as it determines the number of paths one must construct. Consequently, sharper bounds would directly lead to more efficient NPHC methods that use less paths. Moreover, as we shall illustrate, the starting points of the paths are also produced as byproducts of the process for computing upper bounds on the number of solutions. While several general upper bounds exist [31], [32], several studies [26], [33], [34] have hinted that much sharper upper bounds that depend on the topology of the underlying power networks may exist. The main goal of this paper is to establish a significant refinement to these observations using an alternative polynomial formulation of the load flow equations that more naturally bridge the network topology and the theory of Bernshtein-Kushnirenko-Khovanskii bound [35]. This paper is organized as follows: §II formulates algebraic load flow equations and reviews existing results. §III-A describes the tight bound on the number of isolated complex solutions for algebraic load flow equations which will be called the Conjugate Coordinate Bernshtein-Kushnirenko-Khovanskii bound. §III-B propose a novel geometric formulation for the upper bound called adjacency polytope bound. §III-C discusses the computational issues, and §III-D highlights the significant implications of the development of these solution bounds in homotopy methods for solving load flow equations. §IV compares the bounds we will develop with previously known bounds. Necessary but well known concepts from convex geometry and complex algebraic geometry are included in the Appendix for completeness.

[*]Dept. of Mathematics, Michigan State University, East Lansing, MI USA.
[†]Current affiliation: Dept. of Mathematics and Computer Science, Auburn University Montgomery, Montgomery, AL USA.
[‡]Dept. of Applied and Computational Mathematics and Statistics, and Dept. of Chemical and Biomolecular Engineering of University of Notre Dame, Notre Dame, IN USA; and, Centre for the Subatomic Structure of Matter, Dept. of Physics, School of Physical Sciences, University of Adelaide, Adelaide, South Australia, Australia.
[§]Current affiliation: Systems Department, United Technologies Research Center, East Hartford, CT, USA.

II. Algebraic formulation

In this paper, we focus on the mathematical abstraction of a power network which is captured by a graph $G = (B, E)$ and a complex matrix $Y = (Y_{ij})$. Here B is the set of nodes representing "buses", E is the set of edges (a.k.a. branches) representing the connections among buses, and the matrix Y is the nodal *admittance matrix* which assigns a nonzero complex value Y_{ij} (*mutual admittances*) to each edge $(i, j) \in E$. For any $(i, j) \notin E$, $Y_{ij} = Y_{ji} = 0$. Here, Y is *not* assumed to be symmetric, but we require Y_{ij} and Y_{ji} to be both nonzero if $(i, j) \in E$. As a convention, we further require all nodes to be connected with itself to reflect the nonzero diagonal entries Y_{ii} known as *self-admittances*. For brevity, we define n to be the number of non-reference buses (i.e., $|B| = n + 1$) and label the nodes by $0, 1, 2, \ldots, n$. Their complex voltages will be denoted by $v_0, v_1, \ldots, v_n$. Here we fix node 0 to be the designated reference bus for which the voltage v_0 is fixed to a nonzero real constant. In this setup, the load flow equation takes the form of

$$S_i = \sum_{j=0}^{n} Y_{ij}^* v_i v_j^* \quad \text{for } i = 1, \ldots, n, \tag{1}$$

which is a system of n equations in the n variables $v_1, \ldots, v_n$ since v_0, corresponding to the reference bus, is a constant. Here v_i^* and Y_{ij}^* denotes the complex conjugates of v_i and Y_{ij} respectively, and $S_i \in \mathbb{C}$ are the *injected power*. The equations (1) may represent either a transmission or distribution network, with PQ buses. It is the network topology along with other features that can distinguish these cases: a mesh topology would usually correspond to transmission networks, whereas radial topology would correspond to distribution networks.

A solution to (1) is said to be **isolated** if it is the only solution in a sufficiently small neighborhood. Solutions with some $v_k = 0$ are said to be **deficient**. By an application of Sard's Theorem [24], it can be verified that under a generic perturbation of $S_1, \ldots, S_n$, the system (1) has no deficient solutions (deficient solutions may appear for specific choices of $S_1, \ldots, S_n$). We therefore focus only on the non-deficient solutions. The problem central to this paper is counting the isolated non-deficient load flow solutions:

Problem Statement 1. *For a power network, what is the maximum number of isolated non-deficient solutions to* (1)*?*

Following the fruitful algebraic approach taken by works such as [31], [32], we "embed" this problem into a more general algebraic root counting problem: We consider a polynomial system whose solution set captures all the solutions of the above (non-algebraic) system by introducing new variables

$$u_i = v_i^* \quad \text{for each } i = 0, \ldots, n. \tag{2}$$

Substituting them into (1), we obtain algebraic equations

$$S_i = \sum_{j=0}^{n} Y_{ij}^* \, v_i \, u_j \quad \text{for } i = 1, \ldots, n. \tag{3}$$

This is a system of n equations in $2n$ variables. However, a "square" system where the number of variables and equations match is more convenient from an algebraic point of view. We therefore extract n hidden equations by taking the complex conjugates of both sides of each of the above and obtain

$$S_i^* = \sum_{j=0}^{n} (Y_{ij}^* \, v_i \, u_j)^* = \sum_{j=0}^{n} Y_{ij} \, u_i \, v_j \text{ for } i = 1, \ldots, n. \tag{4}$$

We now sever the tie between $\mathbf{u} = (u_1, \ldots, u_n)$ and $\mathbf{v} = (v_1, \ldots, v_n)$ and consider them to be variables independent from one another, i.e., we ignore (2). Then (3) and (4) combine into a system of $2n$ polynomial equations in the $2n$ variables:

$$P_{G,Y,S}(\mathbf{v}, \mathbf{u}) = \begin{cases} \sum_{k=0}^{n} Y_{1k}^* \, v_1 u_k - S_1 = 0 \\ \quad \vdots \\ \sum_{k=0}^{n} Y_{nk}^* v_n u_k - S_n = 0 \\ \sum_{k=0}^{n} Y_{1k} \; u_1 v_k - S_1^* = 0 \\ \quad \vdots \\ \sum_{k=0}^{n} Y_{nk} \, u_n v_k - S_n^* = 0. \end{cases} \tag{5}$$

Here, the values of v_0 and u_0 are fixed, as they correspond to the reference node and are hence constants in the above system. For brevity, this system will be referred to as the **algebraic load flow equations**. This formulation is essentially the algebraic way of rewriting the load flow equations in the "complex conjugate coordinate" which is a common technique in the theory of several complex variables known as *polarization* [36]. It first appeared in [32] to the best of our knowledge. A similar polynomial formulation for a special case was also used in earlier works [31], [37]. It is also employed in the Holomorphic Embedding method [38]. Other polynomial formulations of the load flow equations have also been used (see, e.g., [26]–[28], [39]–[41]).

It is worth noting that in $P_{G,Y,S}$, the topology of the underlying power network is encoded in the set of monomials while entries of Y and S appear as the coefficients. Developing a solution count that exploits network topology via the monomial structure is the main goal of this paper.

Clearly, for every solution $\mathbf{v}$ of the original (non-algebraic) system (1), $P_{G,Y,S}(\mathbf{v}, \mathbf{v}^*) = \mathbf{0}$. That is, $P_{G,Y,S} = \mathbf{0}$ captures all solutions of the original load flow system. In the following, we focus on the algebraic root counting problem:

Problem Statement 2. *For a power network with topology given by a graph G, what is the maximum number of isolated roots of $P_{G,Y,S}$ in $(\mathbb{C} \setminus \{0\})^{2n}$ for all choices of Y and S?*

Here, the "maximum number" means the **lowest upper bound that is also attainable** and shall be distinguished from a mere "upper bound". Of course, the existence of such a "maximum number" is not a priorily guaranteed. One of the goals of this paper is to establish the validity of the above question. Clearly, any answer to Problem 2 provides an upper bound for the answer to Problem 1. It is possible for the algebraic formulation (5) to introduce extraneous solutions (for which $\mathbf{u} \neq \mathbf{v}^*$). This is a reasonable trade-off — with the formulation (5), we get a much easier algebraic system at the expense of potentially introducing extraneous solutions. As a direct consequence of the inherit symmetry in the polarization technique, extraneous solutions must appear in conjugate pairs, $(\mathbf{v}, \mathbf{u})$ and $(\mathbf{u}^*, \mathbf{v}^*)$, as long as $v_0 \in \mathbb{R}$ [36], [39].

Various upper bounds for Problem 2 have been proposed in the past (see [34] for a recent review). The *classical Bézout number* (or **CB number**) provides a simple upper bound. It is a basic fact in algebraic geometry that the number of isolated complex solutions of a polynomial system is bounded above by the CB number. Therefore, for a power network of n (non-reference) buses, and one reference bus, the CB bound is 2^{2n}, since there are $2n$ equations in (5) each of degree 2. A much tighter upper bound on the number of isolated complex solutions, $\binom{2n}{n}$, was derived for the special case of completely interconnected lossless networks by Baillieul and Byrne [31], and the same bound for the general case is established by Li, Sauer, and Yorke [32] (see [39] for a recent alternative derivation of this bound). We shall refer to this bound as the Baillieul-Byrne-Li-Sauer-Yorke bound, or simply **BBLSY bound**. However, neither of these bounds exploit the network topology of a given power system. The link between network topology and complex solution count was first hinted in [33], however, a concrete and computable answer remains elusive.

In a recent study [34], with extensive numerical experiments via the NPHC methods, it was observed that the number of isolated complex solutions is generally significantly lower than both the CB and BBLSY bound for sparsely connected graphs. Based on these observations, it was anticipated that the key to exploiting the network structure of the power system may be to exploit the underlying topology of the power system. In the present work, we show that this maximum number exists and it is given by the *Bernshtein-Kushnirenko-Khovanskii* (or BKK) bound. We then develop a novel approximation of this maximum number, to be called the "adjacency polytope" which has tremendous computational advantage yet is exact in many concrete cases as we shall show.

III. The Maximum Number of Solutions

A. The conjugate coordinate BKK bound

Problem 2 is a special case of the root counting problem for polynomial systems which is an important problem in algebraic geometry that has a wide range of applications [24], [42], [43]. Two basic root counts are provided by the CB and BBLSY bounds described above. One common weakness of the two is that they only utilize the rather incomplete information about the polynomial system — the degree (or "multi-degree"). In the current context, this means that they do not take into consideration the topology of the underlying network. Following up the observations in Ref. [34] we refine these bounds using the theory of BKK bound [35] which accurately captures the network topology of the power systems. Recall that the topological information is encoded in "monomial structure" of (5), i.e., the set of monomials that actually appear. Intuitively, the theory of BKK bound provides a root count in terms of a volume measurement for the geometric shapes spanned by the monomials:

Theorem 1 (Bershtein [35])**.** *Consider the algebraic load flow system of $2n$ polynomial equations* (5) *in $2n$ variables.*

- *(A) The number of isolated solutions the system has in $(\mathbb{C} \setminus \{0\})^{2n}$ is bounded above by the mixed volume of the Newton polytopes for the $2n$ equations.*
- *(B) Without enforcing the conjugate relations among the coefficients, there is an open and dense set of coefficients for which all solutions of the system* (5) *in $(\mathbb{C} \setminus \{0\})^{2n}$ are isolated and the total number is exactly the upper bound given in (A).*

In this, the Newton polytopes are the smallest convex sets that contain the exponent vectors in each equation in (5), and mixed volume is a generalization of volume to a list of geometric bodies that measures the sizes as well as relative orientation of the bodies. The technical definition is included in the Appendix. Here, it is sufficient to take the following interpretation: Part (A) establishes a computable upper bound for the number of isolated solutions that depends on the geometric configuration of the monomial structure (and hence the network topology), and part (B) shows this upper bound is generically exact. The original proof was given in [35]. An alternative proof that gives rise to the development of *polyhedral homotopy* was given in [44]. More detail can be found in standard references such as [25], [45], [46]. In [47], the root count in the above theorem was nicknamed the **BKK bound** after the works of Bernshtein [35], Kushnirenko [48], and Khovanskii [49]. In general, it provides a much tighter bound on the number of isolated zeros of a polynomial system compared to variants of Bézout bounds. More importantly, in the context of load flow equations, the topology of the underlying power network is encoded in the monomial structure, *the BKK bound is therefore topology dependent*.

It is important to note that the "generic exactness" expressed in part (B) of the above theorem only holds when one ignores the tie between Y_{ij} and Y_{ij}^* as well that between S_i and S_i^*. That is, one must allow Y_{ij} and Y_{ij}^* to vary independently in interpreting the above theorem. We shall now bring back the restriction that all the (Y_{ij}, Y_{ij}^*) and (S_i, S_i^*) must be conjugate pairs and investigate the exactness of the BKK bound under these restrictions. We shall fix the sparsity pattern of the Y matrix but allow its entries (and that of S) to vary among the set of nonzero complex numbers. In the following, we shall establish that the BKK bound is always exact for *some* choice of Y and S. In other words, we have the following assertions:

Theorem 2. *Given a graph G, there exist a matrix Y and a vector S for which the number of isolated solutions of the corresponding algebraic load flow equation $P_{G,Y,S} = \mathbf{0}$ in $(\mathbb{C} \setminus \{0\})^{2n}$ is exactly the BKK bound given in Theorem 1.*

Proof. For convenience, let $Z = (Z_1, \ldots, Z_\ell)$ collect all the nonzero entries of Y_{ij} and S_i. That is, Z contains all the nonzero coefficients in (5). Let D be the discriminant provide by part (B) of Theorem 1. We simply have to show that there exists a choice of $Z \in (\mathbb{C} \setminus \{0\})^\ell$ such that the discriminant $D(Z, Z^*) \neq 0$. Suppose no such choice of Z exist, then $D(Z, Z^*) = 0$ for all $Z \in (\mathbb{C} \setminus \{0\})^\ell$. By Lemma 1 in the Appendix, $D(Z, W) = 0$ for all $(Z, W) \in (\mathbb{C} \setminus \{0\})^{2\ell}$. This means D must be a zero polynomial, which is a contradiction. Therefore, we must conclude that there is always a choice of Z (and hence Y and S) such that $D(Z, Z^*) \neq 0$. □

Remark 1. *From the theory of complex variables, an immediate consequence of the above theorem is that the BKK bound must be exact for almost all choices of Y and S. That is, if Y and S are chosen at random (among all complex matrices and vectors of the appropriate sizes) then the probability of picking one for which the BKK bound fails to be exact is zero.*

Alternatively, the generic exactness of BKK bound can also be interpreted in terms of closeness — every choice of (Y, S) is arbitrarily close to some choice for which this bound is exact:

Theorem 3. *Given a graph G, a matrix Y, vector S, and a threshold $\epsilon > 0$, there exists a pair of matrix $\tilde{Y}$ and vector $\tilde{S}$ with $\tilde{Y}$ having the same sparsity pattern as Y and S such that $\|(Y, S) - (\tilde{Y}, \tilde{S})\| < \epsilon$ and the number of isolated solutions in $(\mathbb{C} \setminus \{0\})^{2n}$ of the algebraic load flow equation $P_{G,\tilde{Y},\tilde{S}} = \mathbf{0}$ in* (5) *is exactly the BKK bound given in Theorem 1.*

Proof. As in the previous proof, let $Z = (Z_1, \ldots, Z_\ell)$ collect all the nonzero entries in Y and S. Suppose there exists an open ball Ω of radius ϵ and centered at (Y, S) such that the discriminant $D(Z, Z^*) = 0$ for all $Z \in \Omega$. Then by Lemma 1, $D(Z, W) = 0$ for all $(Z, W) \in \Omega \times \Omega^*$ where $\Omega^* = \{W \mid W^* \in \Omega\}$. That is, D is identically zero on an open domain. A contradiction. Therefore, we must conclude that there must be some $Z = (\tilde{Y}, \tilde{S}) \in \Omega$ for which $D(Z, Z^*) \neq 0$, i.e., the BKK bound is exact for $P_{G,\tilde{Y},\tilde{S}} = \mathbf{0}$. □

Note that the BKK bound given in Theorem 1 depends on the special algebraic formulation given in (5) based on the "conjugate coordinate" (using v_i's and v_i^*'s as variables). When the theory of BKK bound is applied to other algebraic formulations, the actual bound may be different. Moreover, Theorem 2 and 3 may not hold for other formulations. For instance, if (1) is transformed into a polynomial system by using the real and imaginary parts of v_i's as variables (a common practice in power-flow studies as adopted in [26]), then the corresponding BKK bound may not be exact for *any* choice of Y and S. Therefore, to distinguish the BKK bound given in Theorem 1 from similar BKK bound derived from other formulations, we shall call it the **Conjugate Coordinate BKK bound** (or **CCBKK bound**).

B. Solution bound via adjacency polytope

We now develop an approximation of the CCBKK bound that can be analyzed and computed more easily. First, we encode the given graph into a polytope (a geometrical object with flat sides). The definition requires the following notations: Let $\mathbf{e}_0 := \mathbf{0} \in \mathbb{R}^n$, and let $\mathbf{e}_i \in \mathbb{R}^n$ for $i = 1, \ldots, n$ denote the vector that has an entry 1 on the i-th position and zero elsewhere. $(\mathbf{e}_i, \mathbf{e}_j) \in \mathbb{R}^{2n}$ is simply the concatenation of $\mathbf{e}_i, \mathbf{e}_j \in \mathbb{R}^n$. Finally, "conv" denotes the convex hull operator which produces the smallest convex set containing a given set.

Definition 1. *Given an undirected graph $G = (B, E)$, let*

$$\Gamma_G := \bigcup_{(i,j) \in E} \{(\mathbf{e}_i, \mathbf{e}_j)\} \subset \mathbb{R}^{2n}.$$

With this, we define the ***symmetric adjacency polytope*** *to be*

$$\nabla_G := \operatorname{conv}(\Gamma_G \cup \{\mathbf{0}\}).$$

∇_G is a geometric encoding of the power network connectivity with connections manifested as points.

Remark 2. *It is clear that equations in* (5) *always contain many common monomials. Indeed, if (i, j) is an edge, then $v_i u_j$ appear in both the i-th and the $(n+j)$-th equation. That is, the monomial structure of* (5) *has certain level of built-in redundancy. Such redundancy is removed in the construction of ∇_G which involves the union of the set of points representing the edges. In this union common monomials in* (5) *will therefore coalesce into the same point. Consequently, the polytope ∇_G, in a sense, contains much less information than the monomial structure in* (5). *Therefore the encoding ∇_G is advantageous from a computational point of view.*

Theorem 4. *The number of isolated solutions the algebraic load flow system* (5) *has in $(\mathbb{C} \setminus \{0\})^{2n}$ is bounded above by*

$$\mu_G := \operatorname{NVol}_{2n}(\nabla_G)$$

which will be called ***adjacency polytope bound (AP bound)***.

Here "NVol_{2n}" denotes the *normalized volume* in $\mathbb{R}^{2n}$, and it is defined so that the standard "corner simplex" (the corner of a unit hypercube) has volume 1. This definition would guarantee μ_G is always an integer.

Proof. For a nonsingular $2n \times 2n$ matrix M, we can form the new system $M \cdot P_{G,Y,S}$ as the formal matrix-vector product where $P_{G,Y,S}$ is considered as a column vector. This technique is known as *randomization*. Clearly, $M \cdot P_{G,Y,S}(\mathbf{v}, \mathbf{u}) = \mathbf{0}$ if and only if $P_{G,Y,S}(\mathbf{v}, \mathbf{u}) = \mathbf{0}$ and the number of isolated solutions (in $(\mathbb{C} \setminus \{0\})^{2n}$) remains the same under this transformation. It is easy to verify that the *support* of the randomized system $M \cdot P_{G,Y,S}$ is *unmixed* of type $2n$, and the Newton polytope is precisely the symmetric adjacency polytope ∇_G defined in Definition 1. Then by the unmixed form of Bernshtein's Theorem [44], the BKK bound of this randomized system is precisely the normalized volume $\operatorname{NVol}_{2n}(\nabla_G)$. □

Since the CCBKK bound is already shown to be tight (attainable) in Theorem 2 while the AP bound is only shown to be an upper bound, we can immediately conclude that the CCBKK bound is never greater than the AP bound. Moreover, since the initial submission of this paper, the theory of AP bound has been further developed in works such as [50] where the AP bound is shown to be always exactly equal to the CCBKK bound under the mild condition that the power injection $S_1, \ldots, S_n$ are all nonzero:

Proposition 1 (Proposition 3 in [50]). *Given a graph $G = (B, E)$ and nonzero complex constants $S_1, \ldots, S_n$, the CCBKK bound and the AP bound for the induced algebraic load flow system* (5) *are identical.*

Example 1. Consider the simple path graph $G = (\{0, 1, 2, 3\}, E)$ of 4 nodes where each node i is connected to the next node $i + 1$. Recall that we also require each node to have a loop to itself, so the edges in the graph are

$$E = \{(0,0), (0,1), (1,0), (1,1), (1,2), (2,1), (2,2), (2,3), (3,2), (3,3)\}.$$

By Definition 1, the points in Γ_G are therefore $(\mathbf{e}_0, \mathbf{e}_0)$, $(\mathbf{e}_0, \mathbf{e}_1)$, $(\mathbf{e}_1, \mathbf{e}_0)$, $(\mathbf{e}_1, \mathbf{e}_1)$, $(\mathbf{e}_1, \mathbf{e}_2)$, $(\mathbf{e}_2, \mathbf{e}_1)$, $(\mathbf{e}_2, \mathbf{e}_2)$, $(\mathbf{e}_2, \mathbf{e}_3)$, $(\mathbf{e}_3, \mathbf{e}_2)$, and $(\mathbf{e}_3, \mathbf{e}_3)$. With programs for computing volume of convex polytopes to be listed in § III-C, we can easily compute that the AP bound is $\mu_G = \operatorname{NVol}_6(\nabla_G) = \operatorname{NVol}_6(\operatorname{conv} \Gamma_G) = 8$ whereas the BBLSY bound is $\binom{6}{3} = 20$. That is, using the AP bound, we can show that the algebraic load flow equations (5) for such a path graph has at most 8 isolated non-deficient complex solutions, and it is significantly tighter than the existing BBLSY bound.

Though the present contribution focus mainly on the root counting problem for the *algebraic* load flow equations (Problem 2), we shall note that it is possible to have a gap between the root counts for Problem 2 and Problem 1. For instance, using a randomly chosen symmetric Y matrix and S_1, S_2, S_3 that sum to zero, we form the algebraic load flow equations

$$
\begin{aligned}
(0.91 - 0.32i)v_1u_1 + (1.78 - 1.63i)v_1 0.87 + (0.36 - 1.33i)v_1u_2 &= 1.06 + 1.28i \\
(1.90 - 1.19i)v_2u_2 + (0.36 - 1.33i)v_2u_1 + (1.69 - 0.55i)v_2u_3 &= -0.51 - 0.62i \\
(0.44 - 0.64i)v_3u_3 + (1.69 - 0.55i)v_3u2 &= -0.55 - 0.66i \\
(0.91 + 0.32i)u_1v_1 + (1.78 + 1.63i)u_1 0.87 + (0.36 + 1.33i)u_1v_2 &= 1.06 - 1.28i \\
(1.90 + 1.19i)u_2v_2 + (0.36 + 1.33i)u_2v1 + (1.69 + 0.55i)u_2v_3 &= -0.51 + 0.62i \\
(0.44 + 0.64i)u_3v_3 + (1.69 + 0.55i)u_3v2 &= -0.55 + 0.66i
\end{aligned}
$$

induced by the path graph containing 4 nodes. Using numerical solvers for algebraic systems (e.g. Hom4PS-3 [51]), it is easy to verify that this system has only 4 solutions for which $\mathbf{u}^* = \mathbf{v}$. These, of course, correspond to the 4 solutions to the original (non-algebraic) system (1). In addition, two conjugate pairs of extraneous solutions (for which $\mathbf{u} \neq \mathbf{v}^*$) are introduced by the algebraic formulation using conjugate coordinate system (5). The number of extraneous solutions greatly depends on the choice of the coefficients.

It is quite easy to understand how new connection in a power network will change the AP bound: Since the AP bound is formulated in terms of the volume of a polytope which is nondecreasing (i.e., it will either increase or remain unchanged when new points are added), this upper bound must also be nondecreasing when new connections are introduced:

Theorem 5. *For a graph $G = (B, E)$ and two of its nodes i and j that are not directly connected (i.e., $(i, j) \notin E$), let $G' = (B, E \cup \{(i, j)\})$ be the new graph constructed by adding the edge between i and j to G. Then $\mu_G \leq \mu_{G'}$. Moreover, if $\{(\mathbf{e}_i, \mathbf{e}_j), (\mathbf{e}_j, \mathbf{e}_i)\} \subset \nabla_G$ then $\mu_G = \mu_{G'}$.*

Proof. Recall that each edge in a graph contributes certain points (which may or may not be vertices) in the construction of the symmetric adjacency polytope. Since the edges of G is a subset of the edges of G', we can see that $\nabla_G \subseteq \nabla_{G'}$ with the equality hold precisely when the points contributed by (i, j) are already inside ∇_G. With these observations in mind, both parts of this theorem are direct implications of normalized volume being nondecreasing. □

Based on this observation, it can be shown that the AP bound is never more than the BBLSY bound. This is essentially our alternative proof of the BBLSY bound:

Theorem 6. *For a graph $G = (B, E)$, $\mu_G \leq \binom{2(|B|-1)}{|B|-1}$*

Proof. Fixing the set of buses, Theorem 5 states that the AP bound is nondecreasing as new edges are added to the graph. Consequently, the AP bound for any network constructed from this set of buses is bounded above by the AP bound for the graph with most edges, that is, a complete graph. It is easy to verify that for a complete graph $G = (B, E)$ (with loops), $\nabla_G \subseteq (\operatorname{conv} A) + (\operatorname{conv} B)$ where $A = \{\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \ldots \mathbf{e}_n\}$, $B = \{\mathbf{e}_0, \mathbf{e}_{n+1}, \mathbf{e}_{n+2}, \ldots \mathbf{e}_{2n}\}$, and $(\operatorname{conv} A) + (\operatorname{conv} B)$ denotes the Minkowski sum of the two polytopes $(\operatorname{conv} A)$ and $(\operatorname{conv} B)$. Note that both $(\operatorname{conv} A)$ and $(\operatorname{conv} B)$ are n-dimensional. Then by multi-linearity of mixed volume,

$$
\begin{aligned}
&\operatorname{NVol}((\operatorname{conv} A) + (\operatorname{conv} B)) \\
&= \sum_{k=0}^{2n} \binom{2n}{k} \operatorname{MVol}((\operatorname{conv} A)^{(k)}, (\operatorname{conv} B)^{(2n-k)}) \\
&= \binom{2n}{n} \operatorname{MVol}((\operatorname{conv} A)^{(n)}, (\operatorname{conv} B)^{(n)}) = \binom{2n}{n}
\end{aligned}
$$

□

We conclude this section with a reiteration of the various root counts involved in the discussion. Recall that for a power network G and a choice of (Y, S), the number of isolated non-deficient solutions of the original (non-algebraic) load flow equation (1) (physical solutions), the number of isolated non-deficient complex solutions of the algebraic load flow equation (5), and the bounds discussed above are related as follows:

Physical solution count $\leq$ Complex solution count $\leq$ CCBKK bound $\leq$ AP bound $\leq$ BBLSY bound $\leq$ CB bound

Moreover, the CCBKK bound and the AP bound will be identical under the assumption that inject power $S_1, \ldots, S_n$ are all nonzero.

C. Computing CCBKK and AP bounds

The CCBKK bound which is the BKK bound applied to the special "conjugate coordinate" algebraic formulation (5) can be computed using efficient software programs such as DEMiCs [52], Gfan [53], MixedVol [54], MixedVol-2.0 [55]. For larger power networks involving many buses, the induced algebraic load flow equation may contain a large number of terms, and hence parallel computing technology will be essential. MixedVol-3 [56], [57] (with an improved version integrated in Hom4PS-3 [51]) is capable of computing the CCBKK bound for larger power networks in parallel on a wide range of hardware architectures including multi-core systems, NUMA systems, and computer clusters. As noted in Remark 2, however, there is a built-in level of redundancy in the Newton polytopes (see the Appendix) of the algebraic load flow equations. The formulation of the AP bound takes advantage of this natural redundancy and can generally be computed much more easily than the CCBKK bound for larger power networks. The software package libtropicana [58], developed by the first named author, is designed to compute the AP bound for power networks (the normalized volume of the polytopes defined in § III-B). But since the AP bound is formulated in terms of the volume of a convex polytope (the symmetric AP), any software that can compute such volume exactly can be used to provide this bound. A survey on the various algorithms for exact volume computation can be found in [59].

D. Homotopy methods for solving load flow equations

The previous sections described the CCBKK and AP bounds for the number of isolated non-deficient complex solutions to the algebraic load flow equations. It is worth reiterating that the CCBKK bound is more than just an *upper bound*: As shown in Theorem 2 and Remark 1, it is actually the *generic complex root count* for the given network topology in the sense that for almost all choices of Y and S, the total number of isolated non-deficient complex solutions is exactly the CCBKK bound. While the AP bound, in general, may be larger, we shall show in § IV that the two coincide for all the networks we have investigated in the present work. The family of numerical methods known as homotopy methods have been proved to be a robust and efficient approach for solving algebraic load flow equations. One great strength of these methods lies in the parallel scalability: in principle, each solution can be computed independently. This feature is of particular importance in dealing with larger power networks (hence more complicated equations). It is therefore a natural question to ask: is there a homotopy method that can solve (5) by tracking CCBKK bound number of homotopy paths? This section establishes the answer in the affirmative.

This homotopy method is the *polyhedral homotopy* method developed in [44]. Here we briefly state the construction: Choosing a pair of random rational numbers ω_{ij} and ω'_{ij} for $i, j = 1, \ldots, n$. With these we define the *homotopy function*

$$H_{G,Y,S}(\mathbf{v}, \mathbf{u}, t) = \begin{cases} \sum_{k=0}^{n} Y_{1k}(t) v_1 u_k t^{\omega_{1k}} - S_1(t) \\ \vdots \\ \sum_{k=0}^{n} Y_{nk}(t) v_n u_k t^{\omega_{nk}} - S_n(t) \\ \sum_{k=0}^{n} Y'_{1k}(t) u_1 v_k t^{\omega'_{k1}} - S'_1(t) \\ \vdots \\ \sum_{k=0}^{n} Y'_{nk}(t) u_n v_k t^{\omega'_{kn}} - S'_n(t). \end{cases} \tag{6}$$

where

$$Y_{ij}(t) = (1-t) Z_{ij} + t Y^*_{ij} \qquad S_i(t) = (1-t) W_i + t S_i$$
$$Y'_{ij}(t) = (1-t) Z'_{ij} + t Y_{ij} \qquad S'_i(t) = (1-t) W'_i + t S^*_i$$

and (Z_{ij}) and (Z'_{ij}) are randomly chosen complex matrices of the same sparsity structure as Y and $W = (W_i)$ and $W' = (W'_i)$ are two random complex vectors in $\mathbb{C}^n$.

Clearly $H_{G,Y,S}(\mathbf{v}, \mathbf{u}, 1) \equiv P_{G,Y,S}(\mathbf{v}, \mathbf{u})$. For generic choice of Z, Z', W, W', ω and ω', it can be shown that for any $t \in (0, 1)$, the non-deficient solutions of $H_{G,Y,S}(\mathbf{v}, \mathbf{u}, t) = \mathbf{0}$ are all isolated and the total number is exactly the CCBKK bound. Moreover, as t varies in $(0, 1)$, the corresponding solutions of $H_{G,Y,S}(\mathbf{v}, \mathbf{u}, t) = \mathbf{0}$ also vary smoothly forming *solution paths* that collectively reach all the desired solutions of $P_{G,Y,S}(\mathbf{v}, \mathbf{u}) = \mathbf{0}$. Thus, once the "starting points" of each solution path at $t = 0$ are found, standard numerical continuation techniques can be used to track the solution paths and reach *all* the isolated non-deficient complex solutions which would include all the physical solutions (solutions of the original non-algebraic load flow equations). Extraneous solutions (solutions with $v_i \neq u_i$ for some i) can be discarded[1]

An apparent difficulty is in identifying the "starting points". After all, at $t = 0$, $H_{G,Y,S}(\mathbf{u}, \mathbf{v}, t)$ becomes constant. This is surmounted via a construction known as *mixed cells* which are themselves the by-product from computing the CCBKK bound. Here, we refer to standard references [25], [44], [46] for technical details. This method is implemented in Hom4PS-2.0 [60], Hom4PS-3 [51], PHCpack [61], and PHoM [62]. The application of polyhedral homotopy to load flow equations will be explored in future works, here we simply emphasize that with the polyhedral homotopy method, the number of paths one needs to track is precisely the CCBKK bound of (5), and the process of computing this bound also produce the starting points of these paths. This fact adds to the practical importance of a tight bound on the number of isolated solutions to (5): Both the bound itself and its computing process are necessary to kick-start a NPHC method, especially the polyhedral homotopy method, and a tighter bound would directly lead to less search "dead ends".

[1]From a practical point of view, however, one may still want to keep "nearly physical"solutions for which $v_i \approx u_i$ even if the equalities do not hold exactly. This may be of particular importance when coefficients (Y, S) are derived from inaccurate data as it is so often the case for real world power networks.

$\lvert B \rvert$	4	5	6	7	8	9	10	11	12
Solutions	8	16	32	64	128	256	512	1024	2048
CCBKK	8	16	32	64	128	256	512	1024	2048
AP	8	16	32	64	128	256	512	1024	2048
BBLSY	20	70	252	924	3432	12870	48620	184756	705432
CB	64	256	1024	4096	16384	65536	262144	1048576	4194304

Table I: Comparison of the solution bounds for path graphs.

$\lvert B \rvert$	4	5	6	7	8	9	10	11	12
Solutions	16	40	96	224	512	1152	2560	5632	12288
CCBKK	16	40	96	224	512	1152	2560	5632	12288
AP	16	40	96	224	512	1152	2560	5632	12288
BBLSY	20	70	252	924	3432	12870	48620	184756	705432
CB	64	256	1024	4096	16384	65536	262144	1048576	4194304

Table II: Comparison of solution bounds for ring graphs.

IV. Solution bounds for certain power networks

We now provide concrete computation results for CCBKK and AP bounds induced by certain graphs. Recall that all graphs have self-loops for each node, reflecting the nonzero diagonal entries of Y. In all cases, CCBKK and AP bounds are computed via MixedVol-3 [56], [57] and libtropicana [58] respectively. Complex solutions count of specific load flow systems are computed by solving the systems via Hom4PS-3 [51], [63].

A. Path and ring graphs

We first consider two sparse families of graphs — paths and rings (cycles). Table I and Table II show the 5-way comparison among the bounds described above and the actual complex solution count[2] for paths and rings of various sizes. Note that in all cases computed (100 in total, with 10 random Y matrices for each $|B|$), *the actual complex solution count, the CCBKK bound, and the AP bound are exactly the same.* Moreover, for trees, both bounds proposed in this paper seem to grow as 2^n while the best previously known bound, the BBLSY bound, is $\binom{2n}{n}$. The asymptotic advantage is clear since $2^n / \binom{2n}{n} \to 0$ as n grows, and it is clear from the table that the gap between the two can be very large even for small n values (e.g. a 344 fold difference for $n = 11$).

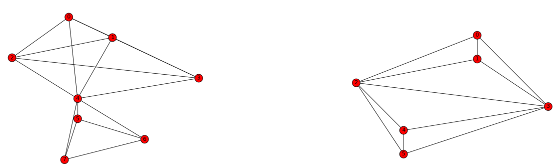
[2]Since the solution count may depend on the coefficients (Y_{ij} and S_i), we used a sample of randomly chosen set of coefficients for each graph.

$c_1 \backslash c_2$	2	3	4	5	6
2	4/4/6	12/12/20	40/40/70	140/140/252	504/504/924
3	12/12/20	36/36/70	120/120/252	420/420/924	1512/1512/3432
4	40/40/70	120/120/252	400/400/924	1400/1400/3432	5040/5040/12870
5	140/140/252	420/420/924	1400/1400/3432	4900/4900/12870	17640/17640/48620
6	504/504/924	1512/1512/3432	5040/5040/12870	17640/17640/48620	63504/63504/184756

Table III: 3-way comparison for two completely connected subnetworks of sizes c_1 and c_2 respectively sharing **one** node.

B. Clusters

Real power networks generally exhibit a level of "clustering" — certain subset of buses are densely connected while on a larger scale, the connections among such subsets are sparse. Here for simplicity, we focus on the most extreme cases where a larger network is created by joining completely connected subnetworks. For comparison, in each case we only show the 3-way comparison among the CCBKK bound, the AP bound, and the BBLSY bound (due to the large amount of data).



(a) Sharing one node (b) Sharing two nodes

Figure 1: Completely connected subgraphs sharing nodes.

1) Subnetworks sharing nodes: See, for example, the networks shown in Figure 1. Table III shows the 3-way comparison for cases where two completely connected subnetworks share *a single (non-reference) bus*. These cases have been studied in [64]. Our computational results agree with their assertion. Table IV shows the similar comparison for cases where two completely connected subnetworks share *two (non-reference) buses*. These cases have been extensively studied in [34] via numerical methods. The results and conjectures in that work are precisely reproduced by our computation. For larger networks, the AP bounds are generally much easier to compute than the CCBKK bound using existing implementations. In Table V and VI, we show the AP bounds these clusters.

$c_1 \backslash c_2$	2	3	4	5	6
2	2/2/2	6/6/6	20/20/20	70/70/70	252/252/252
3	6/6/6	18/18/20	60/60/70	210/210/252	756/756/924
4	20/20/20	60/60/70	200/200/252	700/700/924	2520/2520/3432
5	70/70/70	210/210/252	700/700/924	2450/2450/3432	8820/8820/12870
6	252/252/252	756/756/924	2520/2520/3432	8820/8820/12870	31752/31752/48620

Table IV: 3-way comparison for two completely connected subnetworks of sizes c_1 and c_2 respectively sharing **two** nodes.

2) Completely connected subnetworks connected by edges: For example, Figure 2a shows a network that consists of two cliques of size four and five respectively connected by a single edge. Table VII shows the AP bounds for networks created from joining two completely connected subnetworks by *one* edge. Table VIII shows the AP bounds of the more general cases where the networks consist of multiple completely connected subnetworks of the same sizes connected via edges to form chain-like structure. See, for example, the network shown in Figure 2b where five cliques each of size three are connected via edges that, on a macro level, resembles a chain.

$c_1 \backslash c_2$	2	3	4	5	6	7	8
2	4	12	40	140	504	1848	6864
3	12	36	120	420	1512	5544	20592
4	40	120	400	1400	5040	18480	68640
5	140	420	1400	4900	17640	64680	240240
6	504	1512	5040	17640	63504	232848	864864
7	1848	5544	18480	64680	232848	853776	3171168
8	6864	20592	68640	240240	864864	3171168	11778624

Table V: The AP bounds for graphs consisting of two cliques of size c_1 and c_2 respectively sharing a non-reference node.

$c_1 \backslash c_2$	2	3	4	5	6	7	8
2	2	6	20	70	252	924	3432
3	6	18	60	210	756	2772	10296
4	20	60	200	700	2520	9240	34320
5	70	210	700	2450	8820	32340	120120
6	252	756	2520	8820	31752	116424	432432
7	924	2772	9240	32340	116424	426888	1585584
8	3432	10296	34320	120120	432432	1585584	5889312

Table VI: The AP bounds for graphs consisting of two cliques, of size c_1 and c_2 respectively sharing two non-reference nodes.

V. IEEE 14 BUS SYSTEM

The "IEEE 14-bus system", representing a portion of the power system of the Midwestern USA in the 1960s, is a widely used benchmark system in testing solvers for load flow equations. Here we show that the isolated complex solution count, CCBKK bound, and AP bound are much smaller than previously studied solution bounds. In particular, the CCBKK bound in our formulation of the load flow equations is 427680. This means the polyhedral homotopy method described in §III-D need to trace at most 427680 paths to obtain *all* isolated non-deficient complex solutions. Compared with a previous polynomial formulation [26] which requires the tracking of

Solutions	427680
CCBKK	427680
AP	427680
BKK-MNT	49283072
BBLSY	10400600
CB	67108864

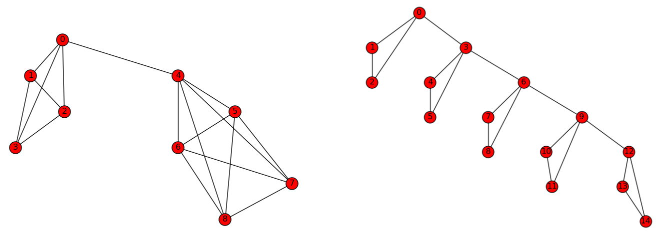
Table IX: Solution bounds for IEEE 14-bus system.

$c_1 \backslash c_2$	1	2	3	4	5	6	7	8	9	10
1		4	12	40	140	504	1848	6864	25740	97240
2	4	8	24	80	280	1008	3696	13728	51480	194480
3	12	24	72	240	840	3024	11088	41184	154440	583440
4	40	80	240	800	2800	10080	36960	137280	514800	1944800
5	140	280	840	2800	9800	35280	129360	480480	1801800	
6	504	1008	3024	10080	35280	127008	465696	1729728		
7	1848	3696	11088	36960	129360	465696	1707552			
8	6864	13728	41184	137280	480480	1729728				
9	25740	51480	154440	514800	1801800					
10	97240	194480	583440	1944800						

Table VII: The AP bound of graphs consisting of two cliques of size c_1 and c_2 joint by a single edge.

$c \backslash m$	1	2	3	4	5	6	7	8
	1	2	3	4	5	6	7	8
1		2	4	8	16	32	64	128
2	2	8	32	128	512	2048	8192	32768
3	6	72	864	10368	124416	1492992	17915904	
4	20	800	32000	1280000				
5	70	9800	1372000					
6	252	127008						
7	924	1707552						
8	3432							

Table VIII: The AP bound for m cliques of size c.



(a) Two completely connected subnetworks jointed via an edge.

(b) A network of several subnetworks linked together via edges.

Figure 2: Completely connected subnetworks linked by edges

49283072 paths (BKK-MNT), our result is around a 115 fold reduction. With both polynomial formulations having the same number of variables and equations, it is reasonable to expect similar reduction in the total time required to track all the paths and hence solve the load flow system. In particular, with a random choice of the Y-matrix, Hom4PS-3 [51], [63] was able to find all solutions in less than 5 minutes (297 seconds) on a single machine with 4 Intel Xeon processors. This example also serves to show the great computational advantage of AP bound over CCBKK bound: Using libtropicana, 77 fold reduction in computation time is achieved in computing the AP bound when compared to the equivalent computation of the CCBKK bound using MixedVol-3 on the same machine.

VI. Conclusion

This paper focused on a tight upper bound on the number of non-deficient complex load flow solutions that take into consideration the network topology (cf. [26]–[29], [31], [32], [34]) which is crucially important in constructing efficient NPHC methods or providing stopping criteria for other iterative methods. We described a specific algebraic formulation of the load flow equations and a corresponding tighter upper bound — the CCBKK bound. We showed that for some graphs there exists at least some generic parameter values for which the CCBKK bound is attainable. Another contribution is the introduction of a novel bound, called adjacency polytope bound, which can be significantly easier to compute for large systems than the CCBKK bound.

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Appendix

For convex polytopes $Q_1, \dots, Q_n \subset \mathbb{R}^n$ and positive numbers $\lambda_1, \dots, \lambda_n$, the n-dimensional volume of the *Minkowski sum* [65] $\lambda_1 Q_1 + \cdots + \lambda_n Q_n$ is a homogeneous polynomial of degree n in $\lambda_1, \dots, \lambda_n$. The coefficient of the term $\lambda_1 \cdots \lambda_n$ in this polynomial is known as the *mixed volume* [65], [66] of $Q_1, \dots, Q_n$, denoted $\operatorname{MVol}(Q_1, \dots, Q_n)$. The BKK bound is formulated in terms of mixed volume. Given a polynomial system $P = (p_1, \dots, p_n)$ the BKK bound is the mixed volume of the Newton polytopes of $p_1, \dots, p_n$.

The proof for Theorem 2 hinges on the polarization lemma in the theory of complex variables:

Lemma 1 (W. Wirtinger)**.** *Suppose that* $H : (\mathbb{C} \setminus \{0\})^n \times (\mathbb{C} \setminus \{0\})^n \to \mathbb{C}$ *is a holomorphic function of the* $2n$ *complex variables* $(\mathbf{z}, \mathbf{w})$*, and that* $H(\mathbf{z}, \mathbf{z}^*) = 0$ *for all* $\mathbf{z} \in (\mathbb{C} \setminus \{0\})^n$*. Then,* $H(\mathbf{z}, \mathbf{w}) = 0$ *for all* $(\mathbf{z}, \mathbf{w}) \in (\mathbb{C} \setminus \{0\})^n \times (\mathbb{C} \setminus \{0\})^n$*.*

The software package libtropicana [58] used to compute the AP bound in the examples shown, is developed by the first named author. For a convex polytope whose vertices have integer coordinates, libtropicana computes its normalized volume by finding a simplicial subdivision. It is based on a pivoting algorithm similar to the core algorithm of lrs [67], [68]. However, unlike lrs, which uses the "reverse search" scheme to optimize memory efficiency, libtropicana is based on a "forward search" scheme that focuses on speed (potentially at the expense of higher memory consumption) for moderate sized polytopes. It is written completely in C++ with optional interface for leveraging BLAS and spBLAS (Sparse BLAS) routines. libtropicana is open source software — its source is freely available under the terms of the LGPL license.

References

[1] P. Kundur, N. J. Balu, and M. G. Lauby, *Power system stability and control*. McGraw-hill New York, 1994, vol. 7.

[2] A. Klos and A. Kerner, "The Non-Uniqueness of Load Flow Solution," in *5th Power Syst. Comput. Conf. (PSCC)*, 1975.

[3] S. Iwamoto and Y. Tamura, "A Load Flow Calculation Method for Ill-Conditioned Power Systems," *IEEE Trans. Power App. Syst.*, vol. PAS-100, no. 4, pp. 1736–1743, Apr. 1981.

[4] J. Glover, M. Sarma, and T. Overbye, *Power System Analysis and Design*. Cengage Learning, 2011.

[5] J. Thorp and S. Naqavi, "Load Flow Fractals," in *IEEE 28th Ann. Conf. Decis. Contr. (CDC)*, Dec. 1989.

[6] D. Bromberg, M. Jereminov, X. Li, G. Hug, and L. Pileggi, "An Equivalent Circuit Formulation of the Power Flow Problem with Current and Voltage State Variables," in *IEEE Eindhoven PowerTech*, Jun. 2015.

[7] A. Trias, "The Holomorphic Embedding Load Flow Method," in *IEEE PES General Meeting*, Jul. 2012.

[8] M. Ilic, "Network Theoretic Conditions for Existence and Uniqueness of Steady State Solutions to Electric Power Circuits," in *IEEE Int. Symp. Circuits Syst. (ISCAS)*, May 1992.

[9] F. Dörfler and F. Bullo, "Novel Insights into Lossless AC and DC Power Flow," in *IEEE PES General Meeting*, Jul. 2013.

[10] K. Dvijotham, S. Low, and M. Chertkov, "Solving the Power Flow Equations: A Monotone Operator Approach," *Submitted. Preprint available: http://arxiv.org/abs/1506.08472*, 2015.

[11] R. Madani, J. Lavaei, and R. Baldick, "Convexification of Power Flow Problem over Arbitrary Networks," *IEEE 54th Ann. Conf. Decis. Contr. (CDC)*, Dec. 2015.

[12] P. Dreesen and B. De Moor, "Polynomial Optimization Problems are Eigenvalue Problems," in *Model-Based Control: Bridging Rigorous Theory and Advanced Technology*, P. M. Hof, C. Scherer, and P. S. Heuberger, Eds. Springer US, 2009, pp. 49–68.

[13] J.-B. Lasserre, *Moments, Positive Polynomials and Their Applications*. Imperial College Press, 2010, vol. 1.

[14] V. Ajjarapu and C. Christy, "The continuation power flow: a tool for steady state voltage stability analysis," *Power Systems, IEEE Transactions on*, vol. 7, no. 1, pp. 416–423, 1992.

[15] W. Ma and S. Thorp, "An Efficient Algorithm to Locate All the Load Flow Solutions," *IEEE Trans. Power Syst.*, vol. 8, no. 3, p. 1077, 1993.

[16] H. Chen, "Cascaded Stalling of Induction Motors in Fault-Induced Delayed Voltage Recovery (FIDVR)," *Masters Thesis, University of Wisconsin-Madison, Department of Electrical and Computer Engineering*, 2011. [Online]. Available: http://minds.wisconsin.edu/handle/1793/53749

[17] D. Molzahn, B. Lesieutre, and H. Chen, "Counterexample to a Continuation-Based Algorithm for Finding All Power Flow Solutions," *IEEE Trans. Power Syst.*, vol. 28, no. 1, pp. 564–565, 2013.

[18] C.-W. Liu, C.-S. Chang, J.-A. Jiang, and G.-H. Yeh, "Toward a CPFLOW-based Algorithm to Compute All the Type-1 Load-Flow Solutions in Electric Power Systems," *IEEE Trans. Circuits Syst. I, Reg. Papers*, vol. 52, no. 3, pp. 625–630, Mar. 2005.

[19] H. Mori and A. Yuihara, "Calculation of Multiple Power Flow Solutions with the Krawczyk Method," in *IEEE Int. Symp. Circuits Syst. (ISCAS)*, 30 May - 2 Jun. 1999.

[20] A. Montes and J. Castro, "Solving the load flow problem using gröbner basis," *ACM SIGSAM Bulletin*, vol. 29, no. 1, pp. 1–13, 1995.

[21] J. Ning, W. Gao, G. Radman, and J. Liu, "The Application of the Groebner Basis Technique in Power Flow Study," in *North American Power Symp. (NAPS)*, Oct. 2009.

[22] H. D. Nguyen and K. S. Turitsyn, "Appearance of multiple stable load flow solutions under power flow reversal conditions," in *PES General Meeting— Conference & Exposition, 2014 IEEE*. IEEE, 2014, pp. 1–5.

[23] D. Cifuentes and P. Parrilo, "Exploiting Chordal Structure in Polynomial Ideals: A Gröbner Bases Approach," *Preprint:* http://arxiv.org/abs/1411.1745, Nov. 2014.

[24] A. J. Sommese and C. W. Wampler, *The numerical solution of systems of polynomials arising in engineering and science*. World Scientific Pub Co Inc, 2005.

[25] T.-Y. Li, "Numerical solution of polynomial systems by homotopy continuation methods," in *Handbook of Numerical Analysis*, P. G. Ciarlet, Ed. North-Holland, 2003, vol. 11, pp. 209–304.

[26] D. Mehta, H. Nguyen, and K. Turitsyn, "Numerical Polynomial Homotopy Continuation Method to Locate All The Power Flow Solutions," *Preprint:* http://arxiv.org/abs/1408.2732, 2014.

[27] S. Chandra, D. Mehta, and A. Chakrabortty, "Exploring the impact of wind penetration on power system equilibrium using a numerical continuation approach," *arXiv preprint arXiv:1409.7844*, 2014.

[28] ——, "Equilibria analysis of power systems using a numerical homotopy method," in *Power & Energy Society General Meeting, 2015 IEEE*. IEEE, 2015, pp. 1–5.

[29] D. Mehta, N. S. Daleo, F. Dörfler, and J. D. Hauenstein, "Algebraic geometrization of the kuramoto model: Equilibria and stability analysis," *Chaos: An Interdisciplinary Journal of Nonlinear Science*, vol. 25, no. 5, p. 053103, 2015.

[30] D. Mehta, D. K. Molzahn, and K. Turitsyn, "Recent advances in computational methods for the power flow equations," *arXiv preprint arXiv:1510.00073*, 2015.

[31] J. Baillieul and C. Byrne, "Geometric Critical Point Analysis of Lossless Power System Models," *IEEE Trans. Circuits Syst.*, vol. 29, no. 11, 1982.

[32] T.-Y. Li, T. Sauer, and J. A. Yorke, "Numerical solution of a class of deficient polynomial systems," *SIAM J. Numer. Anal.*, vol. 24, no. 2, pp. 435–451, Apr. 1987.

[33] S. Guo and F. Salam, "Determining the solutions of the load flow of power systems: Theoretical results and computer implementation," in *, Proceedings of the 29th IEEE Conference on Decision and Control, 1990*, Dec. 1990, pp. 1561–1566 vol.3.

[34] D. K. Molzahn, D. Mehta, and M. Niemerg, "Toward topologically based upper bounds on the number of power flow solutions," *arXiv preprint arXiv:1509.09227*, 2015.

[35] D. N. Bernshtein, "The number of roots of a system of equations," *Funct. Anal. Its Appl.*, vol. 9, no. 3, pp. 183–185, 1975.

[36] W. Wirtinger, "Zur formalen theorie der funktionen von mehr komplexen veränderlichen," *Mathematische Annalen*, vol. 97, no. 1, pp. 357–375, 1927.

[37] J. Baillieul, "The critical point analysis of electric power systems," in *The 23rd IEEE Conference on Decision and Control*, no. 23, 1984, pp. 154–159.

[38] A. Trias, "Fundamentals of the Holomorphic Embedding Load-Flow Method," *arXiv:1509.02421 [cs, math]*, Sep. 2015, arXiv: 1509.02421. [Online]. Available: http://arxiv.org/abs/1509.02421

[39] J. Marecek, T. McCoy, and M. Mevissen, "Power flow as an algebraic system," *arXiv preprint arXiv:1412.8054*, 2014.

[40] T. Chen, J. Marecek, D. Mehta, and M. Niemerg, "A network topology dependent upper bound on the number of equilibria of the kuramoto model," *arXiv preprint arXiv:1603.05905*, 2016.

[41] S. Chandra, D. Mehta, and A. Chakrabortty, "Locating power flow solution space boundaries: A numerical polynomial homotopy approach," *arXiv preprint arXiv:1704.04792*, 2017.

[42] A. Morgan and A. Sommese, "Computing all solutions to polynomial systems using homotopy continuation," *Applied Mathematics and Computation*, vol. 24, no. 2, pp. 115–138, Nov. 1987.

[43] A. P. Morgan, A. J. Sommese, and L. T. Watson, "Finding all isolated solutions to polynomial systems using HOMPACK," *ACM Trans Math Softw*, vol. 15, no. 2, pp. 93–122, Jun. 1989.

[44] B. Huber and B. Sturmfels, "A polyhedral method for solving sparse polynomial systems," *Math. Comput.*, vol. 64, no. 212, pp. 1541–1555, 1995.

[45] ——, "Bernstein's theorem in affine space," *Discrete Comput Geom*, vol. 17, no. 2, pp. 137–141, Mar. 1997.

[46] T. Chen, T.-Y. Li, and X. Wang, "Theoretical aspects of mixed volume computation via mixed subdivision," *Commun. Inf. Syst.*, vol. 14, no. 4, pp. 213–242, 2014.

[47] J. Canny and J. M. Rojas, "An optimal condition for determining the exact number of roots of a polynomial system," in *Proceedings of the 1991 International Symposium on Symbolic and Algebraic Computation*, ser. ISSAC '91. New York, NY, USA: ACM, 1991, pp. 96–102.

[48] A. G. Kushnirenko, "Newton polytopes and the Bezout theorem," *Funct Anal Its Appl*, vol. 10, no. 3, pp. 233–235, Jul. 1976.

[49] A. Khovanskii, "Newton polyhedra and the genus of complete intersections," *Funct. Anal. Its Appl.*, vol. 12, no. 1, pp. 38–46, 1978.

[50] T. Chen, "Unmixing the mixed volume computation," *arXiv preprint arXiv:1703.01684*, 2017.

[51] T. Chen, T.-L. Lee, and T.-Y. Li, "Hom4PS-3: A Parallel Numerical Solver for Systems of Polynomial Equations Based on Polyhedral Homotopy Continuation Methods," in *Mathematical Software – ICMS 2014*, ser. Lecture Notes in Computer Science, H. Hong and C. Yap, Eds. Springer Berlin Heidelberg, Jan. 2014, no. 8592, pp. 183–190.

[52] T. Mizutani and A. Takeda, "DEMiCs: A Software Package for Computing the Mixed Volume Via Dynamic Enumeration of all Mixed Cells," in *Software for Algebraic Geometry*, ser. The IMA Volumes in Mathematics and its Applications, M. Stillman, J. Verschelde, and N. Takayama, Eds. Springer, Jan. 2008, no. 148, pp. 59–79.

[53] A. N. Jensen, "A Presentation of the Gfan Software," in *Mathematical Software - ICMS 2006*, ser. Lecture Notes in Computer Science, A. Iglesias and N. Takayama, Eds. Springer Berlin Heidelberg, Jan. 2006, no. 4151, pp. 222–224.

[54] T. Gao, T.-Y. Li, and M. Wu, "Algorithm 846: MixedVol: a software package for mixed-volume computation," *ACM Trans. Math. Softw. TOMS*, vol. 31, no. 4, pp. 555–560, 2005.

[55] T.-L. Lee and T.-Y. Li, "Mixed volume computation in solving polynomial systems," *Contemp Math*, vol. 556, pp. 97–112, 2011.

[56] T. Chen, T.-L. Lee, and T.-Y. Li, "Mixed volume computation in parallel," *Taiwan. J. Math.*, vol. 18, no. 1, pp. 93–114, 2014.

[57] ——, "Mixed cell computation in Hom4PS-3," *Numer. Algebr. Geom. Spec. Issue J. Symb. Comput.*, (To appear).

[58] T. Chen, "libtropicana: v0.1.1," Jul. 2016. [Online]. Available: https://doi.org/10.5281/zenodo.57133

[59] B. Büeler, A. Enge, and K. Fukuda, "Exact Volume Computation for Polytopes: A Practical Study," in *Polytopes — Combinatorics and Computation*, ser. DMV Seminar, G. Kalai and G. M. Ziegler, Eds. Birkhäuser Basel, Jan. 2000, no. 29, pp. 131–154.

[60] T.-L. Lee, T.-Y. Li, and C.-H. Tsai, "HOM4PS-2.0: a software package for solving polynomial systems by the polyhedral homotopy continuation method," *Computing*, vol. 83, no. 2, pp. 109–133, 2008.

[61] J. Verschelde, "Algorithm 795: PHCpack: A general-purpose solver for polynomial systems by homotopy continuation," *ACM Trans. Math. Softw. TOMS*, vol. 25, no. 2, pp. 251–276, 1999.

[62] T. Gunji, S. Kim, M. Kojima, A. Takeda, K. Fujisawa, and T. Mizutani, "PHoM–a polyhedral homotopy continuation method for polynomial systems," *Computing*, vol. 73, no. 1, pp. 57–77, 2004.

[63] T. Chen, T.-L. Lee, and T.-Y. Li, "Hom4PS-3: an numerical solver for polynomial systems using homotopy continuation methods."

[64] S. Guo and F. Salam, "Determining the Solutions of the Load Flow of Power Systems: Theoretical Results and Computer Implementation," in *IEEE 29th Ann. Conf. Decis. Contr. (CDC)*, Dec. 1990, pp. 1561–1566.

[65] H. Minkowski, "Theorie der Konvexen Körper, insbesondere Begrundung ihres Oberflachenbegriffs," *Gesammelte Abh. Von Hermann Minkowski*, vol. 2, pp. 131–229, 1911.

[66] H. Groemer, "Minkowski addition and mixed volumes," *Geom Dedicata*, vol. 6, no. 2, pp. 141–163, Aug. 1977.

[67] D. Avis and K. Fukuda, "Reverse Search for Enumeration," *Discrete Appl. Math.*, vol. 65, pp. 21–46, 1993.

[68] D. Avis, "A revised implementation of the reverse search vertex enumeration algorithm," in *Polytopes Combinatorics and Computation*, ser. DMV Seminar, G. Kalai and G. M. Ziegler, Eds. Birkhuser Basel, 2000, no. 29, pp. 177–198. [Online]. Available: http://link.springer.com/chapter/10.1007/978-3-0348-8438-9_9