A stratified polyhedral homotopy method for sampling positive dimensional zero sets of polynomial systems[*]

In memory of Professor Tien-Yien Li

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Abstract. Numerical algebraic geometry revolves around the study of solutions to polynomial systems via numerical method. The polyhedral homotopy of Huber and Sturmfels for computing isolated solutions and the concept of witness sets as numerical representations of non-isolated solution components, put forth by Sommese and Wampler, are two of the fundamental tools in this field. In this paper, we show that a modified polyhedral homotopy can reveal sample sets of non-isolated solution components, akin to witness sets, as by-products from the process of computing isolated solutions. In certain cases, this method also leads to a natural decomposition of the BKK bound into a sum of local contributions from individual irreducible components.

Key words. Polyhedral homotopy, witness sets, numerical algebraic geometry

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1. Introduction.

Polynomial systems arise naturally in scientific applications since many computational problems are eventually reduced to algebraic equations. In recent decades, *homotopy methods* emerged as an important class of numerical methods for finding *all* solutions to polynomial systems for their efficiency and scalability [12, 24, 40]. Homotopy methods work by continuously deforming a target system into a starting system that can be solved easily. The corresponding solutions also vary smoothly under this deformation, and they form smooth paths that reach the solutions of the target system. The desired solutions can thus be located by tracking these paths using efficient and robust algorithms.

Among them, the *polyhedral homotopy* of B. Huber and B. Sturmfels [17], developed in the 1990s, is of particular importance due to its ability to optimally exploit combinatorial structures encoded in polynomial systems. Around the same time, the seminal work by A. Sommese and C. Wampler [39] opened up a new frontier in this field by allowing non-isolated (a.k.a. positive-dimensional) solution sets to be computed and manipulated as first-class objects through homotopy methods. In the ensuing years, these two ideas developed separately with minimum interactions with one another.[1] The main goal of this paper is to show these two seemingly independent approaches can be unified into a single numerical method that inherit the strengths of both.

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[1]Notable exceptions include works from the group led by Jan Verschelde [1, 41] in which local Puiseux series representations of positive dimension zero sets are computed through polyhedral-like homotopy methods, as well as an unpublished program by Tsung-Lin Lee for computing individual witness sets using HOM4PS-2.0.

1.1. Combining two homotopy approaches. In this paper, we present a "stratified" version of the polyhedral homotopy for sampling positive-dimensional solution sets of Laurent polynomial systems with the following key advantages

1. The number of paths is the Bernshtein-Kushnirenko-Khovanskii bound, whereas the complexity of the traditional approach is only bounded by the Bézout bounds;
2. This homotopy preserves the monomial structure which is of particular importance in many problems originating from science and engineering where monomial structure imposes additional constraints that are crucial for specific applications; and
3. one single homotopy is used to sample components of all dimensions, including isolated solutions, and sample sets for non-isolated solution components are produced as by-products from the process of computing isolated solutions with minimum overhead.

1.2. Decomposition of BKK bound. Bernshtein's first theorem states that for a system of n Laurent polynomials $(f_1, \dots, f_n)$ in n variables, the number of common isolated zeros in $(\mathbb{C}^*)^n = (\mathbb{C} \setminus \{0\})^n$ is bounded by the mixed volume $\operatorname{mvol}(P_1, \dots, P_n)$ of the Newton polytopes $P_1, \dots, P_n$ of $f_1, \dots, f_n$, respectively. It equals the normalized volume $n! \operatorname{vol}_n(P)$ if $P_1, \dots, P_n = P$ (i.e., the unmixed case, which is established by Kushnirenko). This is known as the Bernshtein-Kushnirenko-Khovanskii (BKK) bound. Indeed, for generic choices of coefficients, all common zeros in $(\mathbb{C}^*)^n$ will be isolated, and this bound will be exact.

However, if the zero set of $(f_1, \dots, f_n)$ in $(\mathbb{C}^*)^n$ contains positive-dimensional components, then the number of isolated zeros in $(\mathbb{C}^*)$ will be strictly less than the BKK bound. A natural question to ask is if it is possible to decompose the BKK bound as a sum of local contributions from each isolated zero and the positive-dimensional components.

This question mirrors the deep question of how to decompose the Bézout number into local contributions from subvarieties that is at the heart of intersection theory. The stratified polyhedral homotopy method proposed in this paper will provide a homotopy-based answer to this question, at least for unmixed cases involving reduced components.

1.3. A motivating example. We start with a simple motivating example.

Example 1.1. Consider a trivial example of a polynomial system $F(x_1, x_2)$, given by

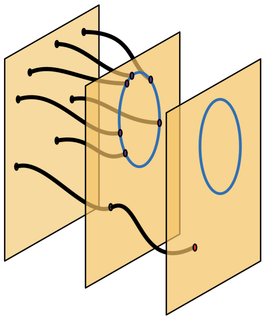
$$\begin{cases} (x_1^2 + x_2^2 - 9)(x_1 + x_2 - 3) = x_1^3 + x_1^2 x_2 - 3x_1^2 + x_1 x_2^2 + x_2^3 - 3x_2^2 - 9x_1 - 9x_2 + 27 \\ (x_1^2 + x_2^2 - 9)(x_1 - x_2 - 1) = x_1^3 - x_1^2 x_2 - 1x_1^2 + x_1 x_2^2 - x_2^3 - 1x_2^2 - 9x_1 + 9x_2 + 9. \end{cases}$$

Its complex zero set consists of two components: A 1-dimensional component V_1 defined by $x_1^2 + x_2^2 - 9 = 0$ (including its distinguished singular points) and a 0-dimensional (i.e., isolated) nonsingular component V_0 at $\mathbf{x}^{(0)} = (x_1, x_2) = (2, 1)$. When the standard polyhedral homotopy method (see Subsection 2.1) is applied, the nonsingular isolated zero $\mathbf{x}^{(0)}$ can be obtained. That is, the polyhedral homotopy defines solution paths, one of which reaches $\mathbf{x}^{(0)}$. With minor modifications, which the rest of this paper will detail, the polyhedral homotopy method can also produce a "numerically well-behaved" sample point from V_1. We consider the "rank-1" perturbation

$$G(x_1, x_2) = \begin{cases} c'_{11} x_1^3 + c'_{12} x_1^2 x_2 + c'_{13} x_1^2 + c'_{14} x_1 x_2^2 + c'_{15} x_2^3 + c'_{16} x_2^2 + c'_{17} x_1 + c'_{18} x_2 + c'_{19} \\ c'_{21} x_1^3 + c'_{22} x_1^2 x_2 + c'_{23} x_1^2 + c'_{24} x_1 x_2^2 + c'_{25} x_2^3 + c'_{26} x_2^2 + c'_{27} x_1 + c'_{28} x_2 + c'_{29}, \end{cases}$$

which is derived from the target system F by replacing the coefficient matrix with

$$\begin{bmatrix} c'_{11} & c'_{12} & c'_{13} & c'_{14} & c'_{15} & c'_{16} & c'_{17} & c'_{18} & c'_{19} \\ c'_{21} & c'_{22} & c'_{23} & c'_{24} & c'_{25} & c'_{26} & c'_{27} & c'_{28} & c'_{29} \end{bmatrix} = \begin{bmatrix} 1 & 1 & -3 & 1 & 1 & -3 & -9 & -9 & 27 \\ 1 & -1 & -1 & 1 & -1 & -1 & -9 & 9 & 9 \end{bmatrix} + \begin{bmatrix} c^*_{11} & c^*_{12} & c^*_{13} & c^*_{14} & c^*_{15} & c^*_{16} & c^*_{17} & c^*_{18} & c^*_{19} \\ c^*_{21} & c^*_{22} & c^*_{23} & c^*_{24} & c^*_{25} & c^*_{26} & c^*_{27} & c^*_{28} & c^*_{29} \end{bmatrix}$$



where $C^* = [c^*_{ij}]$ is a generic complex matrix of rank 1. That is, we modify the coefficient matrix with a generic rank-1 perturbation. Then among the isolated complex zeros of G, at least one is also contained in V_1, the 1-dimensional zero-component defined by F. These zeros depends on the choice of the generic perturbation C^*, but, regardless of the choice, this zero can serve as a "numerically well-behaved" sample point of V_1 in the sense that it will be both a nonsingular zero of G and a smooth point in V_1. We will define a modified polyhedral homotopy $H(x_1, x_2, s)$, which we will call a "stratified" polyhedral homotopy, such that $H(x_1, x_2, \frac{1}{2}) \equiv G(x_1, x_2)$ and $H(x_1, x_2, 0) \equiv F(x_1, x_2)$, and (some of) the solution paths defined by $H(x_1, x_2, s) = \mathbf{0}$ in $\mathbb{C}^2 \times [0, 1]$ will reach sample points in V_1 at $s = \frac{1}{2}$ and the isolated point $\mathbf{x}^{(0)}$ at the end point $s = 0$. In other words, the sample point $\mathbf{x}^{(1)}$ for the 1-dimensional solution component V_1 is produced as a by-product of the process of computing the isolated solution $\mathbf{x}^{(0)}$. The picture on the left shows a cartoonish illustration of the homotopy paths at $s = 0, s = \frac{1}{2}$, and $s = 1$, passing through sample points of V_1 (the blue circle) and the isolated point V_0 (the red point).

1.4. Related works. The approach taken here is closely related to the homotopy method studied by W. Zulehner [44] for finding one point on each connected component of the complex zero set of a polynomial system as well as the stronger version developed by D. Bates, D. Eklund, J. Hauenstein, and C. Peterson [3] that targets the more refined structure known as isosingular set. However, both methods result in complexity measures that are linear in the Bézout number of a given polynomial system, whereas the proposed method has a complexity that is linear in the BKK (Kushnirenko) bound, which can be much lower for sparse systems.

Just like the techniques utilizing "twisted Chow form" and "toric perturbation" developed by M. Rojas [30], the proposed homotopy method also accelerates the computation of positive dimensional zero set by exploiting the combinatorial structure encoded in the Newton polytope of the defining polynomial system. The main difference here is that while Rojas took a resultant-based approach, we are taking a homotopy-based approach.

1.5. Organization. In the rest of this paper, we will describe the construction of this stratified polyhedral homotopy and outline the theoretical underpinnings. To be self-contained, Section 2 will first review notations, concepts, standard results, and theoretical ingredients to be used in the rest of this paper. Section 3 develops the basic construction of a stratified polyhedral homotopy method for sampling positive dimensional solution sets of an unmixed Laurent polynomial system. General cases are considered in Section 4. In Section 5, we explain how this homotopy method can produce, as a by-product, a decomposition of the BKK bound into local contributions from components (including isolated and positive-dimensional components). A few concrete examples are studied in Section 6. We conclude with a few remarks in Section 7. Technical detail of a few well known algorithms for bootstrapping polyhedral homotopy method are included in the appendix (Appendix A) for completeness.

2. Notations and preliminaries.

Let $M_{n\times m}(\mathbb{Z})$ be the set of $n\times m$ integer matrices. A matrix $U\in M_{n\times n}(\mathbb{Z})$ is *unimodular* if $\det U=\pm 1$, in which case $U^{-1}\in M_{n\times n}(\mathbb{Z})$. For $A\in M_{n\times m}(\mathbb{Z})$, there are unimodular $P\in M_{n\times n}(\mathbb{Z})$ and $Q\in M_{m\times m}(\mathbb{Z})$ such that $PAQ=\operatorname{diag}(d_1,\dots,d_r,0,\dots,0)$, where $r=\operatorname{rank} A$, and positive integers $d_1\mid d_2\mid\cdots\mid d_r$ are the *invariant factors* of A. This is the *Smith Normal Form* of A.

For $\mathbf{x}=(x_1,\dots,x_n)$ and $\boldsymbol{\alpha}=(\alpha_1,\dots,\alpha_n)^\top\in\mathbb{Z}^n$, $\mathbf{x}^{\boldsymbol{\alpha}}=x_1^{\alpha_1}\cdots x_n^{\alpha_n}$ is a *Laurent monomial*. Similarly, for $A=\begin{bmatrix}\boldsymbol{\alpha}^{(1)} & \cdots & \boldsymbol{\alpha}^{(m)}\end{bmatrix}\in M_{n\times m}(\mathbb{Z})$ the notation $\mathbf{x}^A=(\mathbf{x}^{\boldsymbol{\alpha}^{(1)}},\dots,\mathbf{x}^{\boldsymbol{\alpha}^{(m)}})$ describes a system of Laurent monomials. It is natural to restrict the domain to the *algebraic torus* $(\mathbb{C}^*)^n=(\mathbb{C}\setminus\{0\})^n$, which has a natural group structure given by componentwise multiplication. A matrix $A\in M_{n\times m}(\mathbb{Z})$ induces a group homomorphism $\mathbf{x}\mapsto\mathbf{x}^A$ from $(\mathbb{C}^*)^n$ to $(\mathbb{C}^*)^m$, which is also complex holomorphic. If $A\in M_{n\times n}(\mathbb{Z})$ is unimodular, then the map $\mathbf{x}\mapsto\mathbf{x}^A$ is an automorphism the group $(\mathbb{C}^*)^n$, and it is also a bi-holomorphic map.

A *Laurent binomial* in $\mathbf{x}=(x_1,\dots,x_n)$ is an expression of the form $c_1\mathbf{x}^{\boldsymbol{\alpha}}+c_2\mathbf{x}^{\boldsymbol{\beta}}$ where $\boldsymbol{\alpha},\boldsymbol{\beta}\in\mathbb{Z}^n$, and $c_1,c_2\in\mathbb{C}^*$. Without altering its zero set in $(\mathbb{C}^*)^n$, the equation $c_1\mathbf{x}^{\boldsymbol{\alpha}}+c_2\mathbf{x}^{\boldsymbol{\beta}}=0$ can be rewritten as $\mathbf{x}^{\boldsymbol{\alpha}-\boldsymbol{\beta}}=-c_2/c_1$. A *Laurent binomial system* is a system of the form $(\mathbf{x}^{\mathbf{a}^1},\dots,\mathbf{x}^{\mathbf{a}^m})=(b_1,\dots,b_m)$ where $\mathbf{a}^i\in\mathbb{Z}^n$ and $b_j\in\mathbb{C}^*$ for $i=1,\dots,n$ and $j=1,\dots,m$. Using the matrix exponent notation, it can be written as $\mathbf{x}^A=\mathbf{b}$ where the integer matrix $A\in M_{n\times m}(\mathbb{Z})$ collects the exponents and the row vector $\mathbf{b}\in(\mathbb{C}^*)^m$ collects all the coefficients.

Lemma 2.1. *For a matrix $A\in M_{n\times n}(\mathbb{Z})$ and any $\mathbf{b}\in(\mathbb{C}^*)^n$, all isolated solutions of Laurent binomial system $\mathbf{x}^A=\mathbf{b}$ are nonsingular, and the total number is $|\det A|$.*

A *Laurent polynomial* is a linear combination of Laurent monomials, i.e., an expression of the form $f=\sum_{k=1}^m c_k\mathbf{x}^{\boldsymbol{\alpha}^{(k)}}$ where each $c_k\in\mathbb{C}^*$ and $\boldsymbol{\alpha}^{(k)}\in\mathbb{Z}^n$. Here, the set $\operatorname{supp}(f):=\{\boldsymbol{\alpha}^1,\dots,\boldsymbol{\alpha}^m\}\subset\mathbb{Z}^n$ is known as the *support* of f. Its convex hull $\operatorname{newt}(f):=\operatorname{conv}(\operatorname{supp}(f))$ is the *Newton polytope* of f. A *Laurent polynomial system* is a system $F=(f_1,\dots,f_q)$ of Laurent polynomials in n variables. Its common zero sets in $(\mathbb{C}^*)^n$ and $\mathbb{C}^n$ are denoted by $\mathcal{V}^*(F)$ and $\mathcal{V}(F)$, respectively. They are equipped with rich structures of *very affine* and *affine* varieties, respectively. If nonempty, they are composed of *irreducible components*, each with a well-defined dimension. The union of their d-dimensional components (isolated zeros) are denoted by $\mathcal{V}_d^*(F)$ and $\mathcal{V}_d(F)$, respectively. Kushnirenko's Theorem and Bernshtein's First Theorem provide us the exact formulae for the maximum number of points in $\mathcal{V}_0^*(F)$.

Theorem 2.2 (Kushnirenko [19]). *For a Laurent polynomial system $F=(f_1,\dots,f_n)$ in $\mathbf{x}=(x_1,\dots,x_n)$ with identical support $S=\operatorname{supp}(f_i)$ for $i=1,\dots,n$, $|\mathcal{V}_0^*(F)|\le n!\operatorname{vol}_n(\operatorname{conv}(S))$.*

Theorem 2.3 (Bernshtein's First Theorem [5]). *For a Laurent polynomial system $F=(f_1,\dots,f_n)$ in the variables $x_1,\dots,x_n$, $|\mathcal{V}_0^*(F)|\le\operatorname{mvol}(\operatorname{newt}(f_1),\dots,\operatorname{newt}(f_n))$.*

Here, $\operatorname{mvol}(P_1,\dots,P_n)$ is the *mixed volume* of the convex polytopes $P_1,\dots,P_n$, and it is defined to be the coefficient of the monomial $\lambda_1\cdots\lambda_n$ in the volume of the *Minkowski sum* $\lambda_1P_1+\cdots+\lambda_nP_n$, which is a homogeneous polynomial in $\lambda_1,\dots,\lambda_n$. The upper bounds given by both theorems are *sharp* in the sense that they hold with equality for generic coefficients. They have since been called the Bernshtein-Kushnirenko-Khovanskii (BKK) bounds.

In the following subsections, we briefly review the four main theoretical ingredients from which we will develop the stratified polyhedral homotopy method. Our review is by no mean comprehensive, and we refer to standard text in this field for thorough exposition.

2.1. Polyhedral homotopy. In their seminal work [17], Huber and Sturmfels introduced the *polyhedral homotopy* method for computing *all* isolated $\mathbb{C}^*$-zeros of Laurent polynomial systems that can optimally exploit their monomial structure.[2]

For a *square* Laurent system $F = (f_1, \dots, f_n)$ in $\mathbf{x} = (x_1, \dots, x_n)$ given by

$$f_i(\mathbf{x}) = \sum_{\mathbf{a} \in S_i} c_{i,\mathbf{a}} \mathbf{x}^{\mathbf{a}}, \quad \text{for } i = 1, \dots, n,$$

we select generic coefficients $c^*_{i,\mathbf{a}}$ for each pair of $i \in \{1, \dots, n\}$ and $\mathbf{a} \in S_i$ and *lifting functions* $\omega_i : S_i \to \mathbb{Q}^+$ with generic images for $i = 1, \dots, n$. Among many variations, the numerically stable formulation for the polyhedral homotopy of Huber and Sturmfels can be described as the homotopy function $H = (h_1, \dots, h_n) : (\mathbb{C}^*)^n \times [0,1]^2 \to \mathbb{C}^n$ given by

$$h_i(\mathbf{x}, t_0, t_1) = \sum_{\mathbf{a} \in S_i} [t_1 c^*_{i,\mathbf{a}} + (1 - t_1) c_{i,\mathbf{a}}] \mathbf{x}^{\mathbf{a}} e^{-M\omega_i(\mathbf{a})t_0} \quad \text{for } i = 1, \dots, n, \tag{2.1}$$

where $M \in \mathbb{R}^+$ is determined by the Newton polytopes of $h_1, \dots, h_n$. This numerically stable variant is different from the original formulation by Huber and Sturmfels [17] and was proposed by S. Kim and M. Kojima [18] and, independently, by T.-L. Lee, T.-Y. Li, and C.-H. Tsai [21]. This formulation will be referred to as *the classical polyhedral homotopy*.

Clearly, H is continuous and $H(\mathbf{x}, 0, 0) \equiv F(\mathbf{x})$. Moreover, along any given smooth path $(t_0(s), t_1(s))$ in the parameter space $(0,1)^2$, under the genericity assumption, the isolated $\mathbb{C}^*$-zero of $H(\mathbf{x}, t_0, t_1)$ also vary smoothly and form "solution paths". The limit points of these solution paths as $(t_0, t_1) \to (0, 0)$ reach *all* isolated $\mathbb{C}^*$-zeros of F. The starting points of these solution paths can be computed by solving a series of Laurent binomial systems. These binomial systems are, in turn, derived from a process known as mixed cell computation.

Once these starting points are obtained, the corresponding solution paths can be tracked via standard numerical algorithms, known as "path trackers", toward their end points, which include all isolated $\mathbb{C}^*$-zeros of the target system F.

In this formulation, there is some flexibility in choosing the parameter path $(t_0(s), t_1(s))$. One choice that is widely adopted in recent implementations is the path $(t_0(s), t_1(s)) = (s, s)$. In contrast, the "2-step" procedure takes the piecewise linear path $(1, 1) \to (0, 1) \to (0, 0)$.

2.2. Parameter homotopy. The smoothness of the solution paths defined by the homotopy (2.1) over a parameter path $\mathbf{t}(s)$ and their ability to reach *all* isolated $\mathbb{C}^*$-zeros of the target system F are the key features that make this homotopy method practical. Indeed, much of the work in the field of numerical homotopy continuation methods are devoted to the rigorous proof of these two properties (nicknamed "smoothness" and "accessibility" properties in Ref. [25]) for various homotopy constructions. One general result that will be referenced repeatedly is the Parameter Homotopy Theorem of A. Morgan and A. Sommese [29] for homotopy constructions of the form $H(\mathbf{x}, t) = F(\mathbf{x}; \mathbf{p}(t))$ where the coefficients of a polynomial system F are polynomial functions in complex parameters $\mathbf{p} = (p_1, \dots, p_m)$.

[2] In a parallel development, a recursive homotopy method that can also take advantage of the Newton polytope structure to solve Laurent polynomial systems was proposed by J. Verschelde, P. Verlinden, and R. Cools around the same time [42]. This recursive homotopy method has also been referenced as *polyhedral homotopy* in some papers. The present paper, however, only focuses on extending the polyhedral homotopy method of B. Huber and B. Sturmfels [17].

Theorem 2.4 (Parameter homotopy ([40] Theorem 7.1.1, [29])). *Let $F(\mathbf{x};\mathbf{p})$ be a system of n polynomials in the n variables $\mathbf{x} = (x_1, \ldots, x_n)$ and m parameters $\mathbf{p} = (p_1, \ldots, p_m)$, and let $\mathcal{N}(\mathbf{p})$ be the number of (isolated) nonsingular zeros of $F(\mathbf{x};\mathbf{p})$ in $\mathbb{C}^n$ for a given $\mathbf{p}$. Then,*

1. *$\mathcal{N}(\mathbf{p})$ is finite, and it is the same, say $\mathcal{N}$, for almost all $\mathbf{p} \in \mathbb{C}^m$;*
2. *For all $\mathbf{p} \in \mathbb{C}^m$, $\mathcal{N}(\mathbf{p}) < \mathcal{N}$;*
3. *The subset of $\mathbb{C}^m$ where $\mathcal{N}(\mathbf{p}) = \mathcal{N}$ is Zariski open (and nonempty), i.e., the exceptional set $P^* = \{\mathbf{p} \in \mathbb{C}^m \mid \mathcal{N}(\mathbf{p}) < \mathcal{N}\}$ is an affine algebraic set contained within an algebraic set of dimension $n-1$.*
4. *The homotopy $F(\mathbf{x};\mathbf{p}(t)) = 0$ with an analytic function $\mathbf{p}(t) : [0,1] \to \mathbb{C}^m \setminus P^*$ has $\mathcal{N}$ continuous and nonsingular solution paths;*
5. *As $t \to 0$, the limits of the solution paths of the homotopy $F(\mathbf{x};\mathbf{p}(t)) = 0$ with $\mathbf{p}(t) : (0,1] \to \mathbb{C}^m \setminus P^*$ include all the (isolated) nonsingular zeros of $F(\mathbf{x};\mathbf{p}(0)) = 0$ in $\mathbb{C}^n$.*

The variation with $\mathbf{p}$ being the coefficients in F, including constants, was also discovered by T.-Y. Li, T. Sauer, and J. Yorke [26], which led to the extension of the BKK bound [27, 32].

2.3. Positive dimensional zero sets and witness sets. In their pioneer work [39], A. Sommese and C. Wampler kick-started the development of *numerical algebraic geometry*, a new field in computational mathematics that focuses on the study of positive-dimensional solution sets defined by polynomial systems, i.e., *algebraic sets*, via numerical homotopy methods. (See Refs. [16, 38] for an accessible survey and a broad overview of the field, respectively) One of the fundamental building block in field is the concept of "linear slices". A linear slice of a solution set is its intersection with an affine subspace, which can help reveal important structural information about the solution set itself.

Theorem 2.5 (Linear Slicing ([40] Theorem 13.2.1)). *Let $V \subset \mathbb{C}^m$ be a pure d-dimensional algebraic set. There is a Zariski open dense $U \subset \mathbb{P}^m$ such that for $\mathbf{c} \in U$ and $L = \mathcal{V}(\mathcal{L}(\mathbf{z};\mathbf{c}))$,*

1. *if $d = 0$, then $L \cap V$ is empty;*
2. *if $d > 0$, then $L \cap V$ is nonempty and $(d-1)$-dimensional,*
3. *if $d > 1$ and V is irreducible, then $L \cap V$ is irreducible.*

Here, the defining equations of hyperplanes in $\mathbb{C}^m$ are parametrized by points in the complex projective space $\mathbb{CP}^m$, through the map

$$[c_0 : c_1 : \cdots : c_m] \mapsto \mathcal{L}(z_1, \ldots, z_m; c_0, c_1, \ldots, c_m) = c_0 + c_1 z_1 + \cdots + c_m z_m.$$

The stronger version needed in this paper allows for systems of linear polynomials used as linear slicing equations, which we restate here.

Proposition 2.6 ([40] Theorem 13.2.2 and Lemma 13.2.3). *Let $V \subset \mathbb{C}^m$ be a pure d-dimensional affine algebraic set with $d \geq 1$. There is a Zariski open dense subset $U \subset (\mathbb{C}^{(m+1)})^k$ such that for $(\mathbf{c}_1^*, \ldots \mathbf{c}_k^*) \in U$ and $L = \mathcal{V}(\mathcal{L}(\mathbf{z};\mathbf{c}_1^*, \ldots, \mathbf{c}_k^*))$,*

1. *if $d < k$, then $L \cap V$ is empty;*
2. *if $d > k$, then $L \cap V$ is nonempty and positive-dimensional,*
3. *if $d = k$, then $L \cap V$ is nonempty and 0-dimensional.*

Moreover, if V is a component of the zero set of a polynomial system F of multiplicity 1, then $L \cap V$ is a component of $\mathcal{V}(F, \mathcal{L}(\mathbf{z};\mathbf{c}_1^*, \ldots, \mathbf{c}_k^*))$ of multiplicity 1.

The last case in the list above is of particular importance, and it leads to the concept of *witness set* [35, 39] that has its theoretical underpinning in the rich classical study of the connections between algebraic sets and their linear sections [4].

Remark 2.7. The linear slices in this proposition are simply parametrized by k-tuples of complex vectors $C = (\mathbf{c}_1^*, \ldots, \mathbf{c}_k^*)$. As noted in Ref. [40], this is a rather coarse parametrization since the image of C under any nonsingular linear transformation would result in the same linear slicing. The much more natural parameter space is the Grassmannian $\mathrm{Gr}(k, \mathbb{C}^n)$. This distinction, however, is not important in our discussion, and we will prefer the parametrization using k-tuples of complex vectors since they can be chosen at random directly.

2.4. Randomization. The final ingredient is the "randomization" process. For a system F of q Laurent polynomials and a $k \times q$ matrix Λ, every zero of F is, of course, a zero of $\Lambda \cdot F$, if F is considered as a column vector. The following result provides the complete description of the connection between the zero sets of F and $\Lambda \cdot F$, respectively, for generic choices of Λ.

Theorem 2.8 ([40] Theorem 13.5.1). *Let $F = (f_1, \ldots, f_q)$ be a system of polynomials on $\mathbb{C}^n$. Assume $V \subset \mathbb{C}^n$ is an irreducible affine algebraic set. Then there is a nonempty Zariski open set U of $k \times q$ matrices such that for all $\Lambda \in U$,*

1. *if $\dim V > n - k$, then V is an irreducible component of $\mathcal{V}(F)$ if and only if it is an irreducible component of $\mathcal{V}(\Lambda \cdot F)$;*
2. *if $\dim V = n - k$, then V is an irreducible component of $\mathcal{V}(F)$ implies that V is also an irreducible component of $\mathcal{V}(\Lambda \cdot F)$; and*
3. *if V is an irreducible component of $V(F)$, its multiplicity as a solution component of $\Lambda \cdot F(\mathbf{x}) = \mathbf{0}$ is greater than or equal to its multiplicity as a solution component of $F(\mathbf{x}) = \mathbf{0}$, with equality if either multiplicity is 1.*

This produces a particularly useful preprocessing step for solving overdetermined polynomial systems. Any system of q polynomials $F = (f_1, \ldots, f_q)$ in n variables with $q > n$ can be converted into a *square* system $\Lambda \cdot F$ of n polynomials in n variables through an $n \times q$ nonsingular matrix Λ. Every zero of F will be a zero of $\Lambda \cdot F$.

3. Stratified polyhedral homotopy for standard unmixed cases. Based on the four ingredients reviewed above, this section aims to develop a homotopy continuation algorithm, in the spirit of the cascade method [35], for numerically sampling reduced irreducible components of all dimensions of the zero set of a Laurent polynomial system $F = (f_1, \ldots, f_q)$ in the variables $x_1, \ldots, x_n$. Here, a *reduced* irreducible component of $\mathcal{V}^*(F)$ is simply an irreducible component of multiplicity 1. They are also referred to as *generically reduced* irreducible component since at almost all points on such a component, the nullity of the Jacobian matrix DF equals the dimension of the component.

The goal is to construct a homotopy function $H(\mathbf{x}, s)$ such that its zero set $\{(\mathbf{x}, s) \in (\mathbb{C}^*)^n \times (0, 1] \mid H(\mathbf{x}, s) = \mathbf{0}\}$ consists of piecewise smooth solution paths that will pass through finite "sample sets" $V_n, V_{n-1}, \ldots, V_1, V_0$ with $V_d \subset \mathcal{V}_d^*(F)$ and $V_d = \varnothing$ if and only if $V_d^*(F) = \varnothing$ for $d = n, n-1, \ldots, 0$. Moreover, for each reduced irreducible component of $\mathcal{V}_d^*(F)$, V_d contains at least one nonsingular point of that component. In other words, the homotopy H defines homotopy paths that can sample every reduced irreducible component of $\mathcal{V}^*(F)$.

For simplicity, we first focus on a family of unmixed Laurent systems for which the construction of the proposed homotopy has a straightforward geometric interpretation. This family will be referred to as the "standard unmixed cases", which we shall define below. More general cases will be discussed in Section 4.

Recall that a Laurent system $F = (f_1, \ldots, f_q)$ is *unmixed* if the supports $\operatorname{supp}(f_i)$, for $i = 1, \ldots, q$, are all identical. In this case, this common support is denoted $\operatorname{supp}(F)$. We can express such an unmixed Laurent polynomial system in n variables in the compact notation

$$F(x_1, \ldots, x_n) = F(\mathbf{x}) = \begin{cases} f_1(\mathbf{x}) = \mathbf{c}_1 \cdot \mathbf{x}^A \\ \qquad \vdots \\ f_q(\mathbf{x}) = \mathbf{c}_q \cdot \mathbf{x}^A, \end{cases} \tag{3.1}$$

where the support matrix $A = [\,\mathbf{a}_1 \ \cdots \ \mathbf{a}_m\,] \in M_{n\times m}(\mathbb{Z})$, with $m = |\operatorname{supp}(F)| > 0$, collects the exponent vectors in $\operatorname{supp}(F)$ as columns, $\mathbf{c}_k$'s are row vectors collecting corresponding coefficients, and $\mathbf{c}_k \cdot \mathbf{x}^A$ denotes the dot product between the two row vectors. To further simplify our constructions, we first restrict our attention to systems in a "standard form".

Definition 3.1. *The unmixed Laurent polynomial system in* (3.1) *is said to be in* ***standard form*** *if the support matrix* $A \in M_{n\times m}(\mathbb{Z})$ *has the following properties*

1. $m > n + 1$ *;*
2. *A has a zero column ;*
3. *A has full row-rank ;*
4. *The invariant factors of A are ± 1 .*

These conditions can be assumed without loosing much generality: Condition 1 simply eliminate simpler systems for which the proposed method would be unnecessary. Indeed, if $m \le n + 1$, then the $\mathbb{C}^*$-zero set of F is either empty or defined by binomials, and much simpler methods can be used to describe the zero sets. Condition 2 is the requirement that each Laurent polynomial has nonzero constant term, and it can be satisfied by multiplying each polynomial by a Laurent monomial without altering the $\mathbb{C}^*$-zero set of F. Condition 3 ensures that there is no nontrivial toric actions on the $\mathbb{C}^*$-zero set when generic coefficients are used. If $r := \operatorname{rank}(A) < n$, then every $\mathbb{C}^*$-zero $\mathbf{x}$ of F belong to a toric orbit of zeros parametrized by a $\mathbb{C}^*$-valued function $\mathbf{t} \mapsto \mathbf{x} \circ \mathbf{t}^{\mathbf{v}}$ defined on $(\mathbb{C}^*)^r$, where $\mathbf{v}$ is a primitive generator of the left kernel of A. In that case, the $\mathbb{C}^*$-zero set of F can be projected down to $(\mathbb{C}^*)^r$ so that it is defined by an unmixed Laurent system that satisfies this condition. Finally, condition 4, i.e. the torsion-free condition, greatly simplifies our discussions, and Subsection 4.1 will describe the procedure that will reduce the general case to the torsion-free case. For now, we restrict our attention to the standard form.

3.1. Laurent polynomial systems as linear slices. In this paper, we aim to show the seemingly independent approaches of the classical polyhedral homotopy and the linear slicing method from numerical algebraic geometry can be unified into a single numerical method. In service of this goal, we first establish a proper viewpoint through which we can see both. In particular, we will make repeated use of the key observation that under the above assumptions, the zero set $\mathcal{V}^*(F)$ of the unmixed system (3.1) in standard form can be considered as a linear slicing on a binomial system.

Lemma 3.2. *Let $A \in M_{n\times m}(\mathbb{Z})$ be the support matrix of the unmixed system (3.1) in standard form. Then there exists a matrix $B \in M_{m\times m-n}(\mathbb{Z})$ for which $\mathcal{V}^*(F)$ is the image, under the bi-holomorphic map $\phi_A(\mathbf{x}) = \mathbf{x}^A$, of the $\mathbb{C}^*$-zero set of the Laurent system*

$$G(z_1,\dots,z_m) = G(\mathbf{z}) = \begin{cases} \mathbf{z}^B - \mathbf{1} = \mathbf{0} \\ \quad \mathbf{c}_i \cdot \mathbf{z} = 0 \quad \text{for } i = 1,\dots,q \end{cases}$$

where $\mathbf{c}_1,\dots,\mathbf{c}_q$ are the coefficient vectors of the original polynomial system (3.1).

This is the basic setup for the "A-philosophy" for Laurent systems consolidated in the classical text by I. Gel'fand, M. Kapranov, and A. Zelevinsky [15]. We include an elementary and constructive proof for later reference.

Proof. Under the assumption that A is of full row rank and has invariant factors ± 1, there are unimodular matrices $P \in M_{n\times n}(\mathbb{Z})$ and $Q \in M_{m\times m}(\mathbb{Z})$ such that $PAQ = \begin{bmatrix} I_n & \mathbf{0}_{n\times k}\end{bmatrix}$, where $k = m-n > 0$. Let $B \in M_{m\times k}(\mathbb{Z})$ be the rightmost k columns of Q, which spans ker A, and let $C \in M_{k\times m}(\mathbb{Z})$ be the bottommost k rows of Q^{-1}. Then $CB = I_k$, and hence

$$\begin{bmatrix} C \\ A \end{bmatrix} B = \begin{bmatrix} CB \\ AB \end{bmatrix} = \begin{bmatrix} I \\ \mathbf{0} \end{bmatrix}.$$

Our assumptions ensure that $\left[\begin{smallmatrix} C \\ A \end{smallmatrix}\right]$ is unimodular since

$$\begin{bmatrix} I & \\ & P \end{bmatrix} \begin{bmatrix} C \\ A \end{bmatrix} Q = \begin{bmatrix} CQ \\ PAQ \end{bmatrix} = \begin{bmatrix} \mathbf{0} & I_k \\ I_n & \mathbf{0} \end{bmatrix}.$$

Therefore, $\left[\begin{smallmatrix} C \\ A \end{smallmatrix}\right]^{-1} \in M_{m\times m}(\mathbb{Z})$. Let $T = \mathcal{V}^*(\mathbf{z}^B - \mathbf{1}) \subset (\mathbb{C}^*)^m$. We shall construct a bi-holomorphic map between points in $\mathcal{V}^*(F)$ and points in a linear slice of T. Consider the map $\phi_A : (\mathbb{C}^*)^n \to (\mathbb{C}^*)^m$ given by $\phi_A(\mathbf{x}) := \mathbf{x}^A$. For any $\mathbf{x} \in (\mathbb{C}^*)^n$, $(\phi_A(\mathbf{x}))^B = \mathbf{x}^{AB} = \mathbf{x}^{\mathbf{0}} = \mathbf{1}$, and thus $\phi_A(\mathbf{x}) \in T$. I.e., $\phi_A((\mathbb{C}^*)^n) \subseteq T$. It remains to show that the restriction of ϕ on T is bi-holomorphic. Define $\psi : T \to (\mathbb{C}^*)^m$ given by $\psi(\mathbf{z}) = \mathbf{z}^{\left[\begin{smallmatrix} C \\ A \end{smallmatrix}\right]^{-1}}$. For any $\mathbf{z} \in T$, write $\psi(\mathbf{z})$ as $[\mathbf{y}\ \mathbf{x}]$ with $\mathbf{x} \in (\mathbb{C}^*)^n$ and $\mathbf{y} \in (\mathbb{C}^*)^k$, then by construction $\mathbf{z} = \psi(\mathbf{z})^{\left[\begin{smallmatrix} C \\ A \end{smallmatrix}\right]}$, and hence

$$\mathbf{1} = \mathbf{z}^B = (\psi(\mathbf{z}))^{\left[\begin{smallmatrix} C \\ A \end{smallmatrix}\right] B} = \begin{bmatrix} \mathbf{y} & \mathbf{x} \end{bmatrix}^{\left[\begin{smallmatrix} I \\ \mathbf{0} \end{smallmatrix}\right]} = \mathbf{y} \circ \mathbf{1} = \mathbf{y}.$$

Therefore,

$$\mathbf{z} = \psi(\mathbf{z})^{\left[\begin{smallmatrix} C \\ A \end{smallmatrix}\right]} = \begin{bmatrix} \mathbf{1} & \mathbf{x} \end{bmatrix}^{\left[\begin{smallmatrix} C \\ A \end{smallmatrix}\right]} = \mathbf{1}^C \circ \mathbf{x}^A = \mathbf{x}^A.$$

Let $\pi : (\mathbb{C}^*)^m \to (\mathbb{C}^*)^n$ be the projection to the last n coordinates, then for any $\mathbf{z} \in T$,

$$\phi(\pi(\psi(\mathbf{z}))) = \phi(\mathbf{x}) = \mathbf{x}^A = \mathbf{z}.$$

Conversely, for any $\mathbf{x} \in (\mathbb{C}^*)^n$,

$$\pi(\psi(\phi(\mathbf{x}))) = \pi(\psi(\mathbf{x}^A)) = \pi\left(\mathbf{x}^{A\left[\begin{smallmatrix} C \\ A \end{smallmatrix}\right]^{-1}}\right) = \pi\left(\mathbf{x}^{[\mathbf{0}\ I]}\right) = \pi\left(\begin{bmatrix} \mathbf{1} & \mathbf{x} \end{bmatrix}\right) = \mathbf{x}.$$

Therefore, the composition $\pi \circ \psi : T \to (\mathbb{C}^*)^n$ is the inverse of the restriction of ϕ onto T as shown in the following commutative diagram:

$$\begin{tikzcd} (\mathbb{C}^*)^m \arrow[d, "\pi"'] & T \arrow[l, "\psi"'] \\ (\mathbb{C}^*)^n \arrow[ur, "\phi"'] & \end{tikzcd}$$

Moreover, since both ϕ and $\pi \circ \psi$ are given by Laurent monomial maps, the restriction $\phi : (\mathbb{C}^*)^n \to T$ is a (bijective) bi-holomorphic map. Hence, we have the bi-holomorphic correspondence between the $\mathbb{C}^*$-zero sets:

$$\mathbf{c}_i \cdot \mathbf{x}^A = 0 \quad \text{for } i = 1, \dots, q \qquad \iff \qquad \begin{cases} \mathbf{z}^B - \mathbf{1} = \mathbf{0} \\ \quad \mathbf{c}_i \cdot \mathbf{z} = \mathbf{0} \quad \text{for } i = 1, \dots, q. \end{cases}$$

as claimed. ■

3.2. Toric slicing formulation. From the view point of the above lemma, irreducible components in $\mathcal{V}^*(F)$ can be sampled through toric versions of linear slices.

Definition 3.3. *Given an unmixed Laurent polynomial system in standard form, as given in (3.1), in $\mathbf{x} = (x_1, \dots, x_n)$, a nonnegative integer $d \le n$, and vectors $\mathbf{c}_1^*, \dots, \mathbf{c}_d^* \in (\mathbb{C}^*)^m$, we define the corresponding* ***rank** d **toric slicing system*** *to be*

$$F^{(d)}(\mathbf{x}) = \begin{cases} \mathbf{c}_k \cdot \mathbf{x}^A & \text{for } k = 1, \dots, q \\ \mathbf{c}_k^* \cdot \mathbf{x}^A & \text{for } k = 1, \dots, d \end{cases} \tag{3.2}$$

The set $\mathcal{V}_0^*(F^{(d)}) \subset \mathcal{V}^*(F)$ will be called a ***rank** d **sample set*** *of F.*

Note that this definition implicitly depends on the choice of the vectors $\mathbf{c}_1^*, \dots, \mathbf{c}_d^* \in (\mathbb{C}^*)^m$. However, this dependence is of little interest here since the choice is always assumed to be generic in our discussions.

Lemma 3.4. *Let F, given in (3.1), be an unmixed Laurent polynomial system in standard form. If $\mathcal{V}_d^*(F)$ is nonempty and reduced for some nonnegative integer $d < n$, then there is a nonempty Zariski open set $U \subseteq ((\mathbb{C}^*)^m)^d$ such that for all $(\mathbf{c}_1^*, \dots, \mathbf{c}_d^*) \in U$, $\mathcal{V}_0^*(F^{(d)})$ consists of finitely many nonsingular points, and all these points are in $\mathcal{V}_d^*(F)$.*

Proof. By Lemma 3.2, there is a matrix $B \in M_{m \times (m-n)}(\mathbb{Z})$ such that the $\mathcal{V}^*(F) \subset (\mathbb{C}^*)$ is bi-holomorphically equivalent to $V' = \mathcal{V}^*(\mathbf{z}^B - \mathbf{1}, \mathcal{L}(\mathbf{z}; \mathbf{c}_1, \dots, \mathbf{c}_q)) \subset (\mathbb{C}^*)^m$. Then each d-dimensional irreducible component of $\mathcal{V}^*(F)$ corresponds to a d-dimensional irreducible component of V'. Under the same bi-holomorphic map, $\mathcal{V}^*(F^{(d)})$ is equivalent to $\mathcal{V}^*(G^{(d)})$, where

$$G^{(d)}(\mathbf{z}) = \begin{cases} \mathbf{z}^B - \mathbf{1} \\ \mathbf{c}_k \cdot \mathbf{z} & \text{for } k = 1, \dots, q \\ \mathbf{c}_k^* \cdot \mathbf{z} & \text{for } k = 1, \dots, d. \end{cases}$$

Note that $\mathcal{V}^*(G^{(d)})$ is precisely the linear slice of $\mathcal{V}^*(G)$ with respect to $\mathcal{L}(\mathbf{z}; \mathbf{c}_1^*, \dots, \mathbf{c}_d^*)$. By the Linear Slicing Theorem (Theorem 2.5 and Proposition 2.6), the isolated zeros of $G^{(d)}$ in $(\mathbb{C}^*)^n$ are all nonsingular and are contained in the d-dimensional components of $\mathcal{V}^*(G)$. ■

In general, if the requirement for $\mathcal{V}_d^*(F)$ to be reduced is dropped, $\mathcal{V}_0^*(F^{(d)})$ may contain singular (non-smooth) points, i.e., points where rank $DF^{(d)} < n$. Yet, by restriction, the above constructions can still be applied to each individual reduced irreducible component of $\mathcal{V}_d^*(F)$.

Corollary 3.5. *Let V be a nonempty and reduced irreducible d-dimensional component of $\mathcal{V}^*(F)$, then there is a nonempty Zariski open set $U \subseteq (\mathbb{C}^*)^{m\times d}$ such that for all $(\mathbf{c}_1^*, \dots, \mathbf{c}_d^*) \in U$, $\mathcal{V}_0^*(F^{(d)}) \cap V$ is nonempty, and it consists of finitely many nonsingular points in V.*

These lemmas justified that a rank d sample set of F is indeed a sample set for each reduced d-dimensional irreducible component of $\mathcal{V}^*(F)$. The subsections that follow aim to set up an efficient homotopy method for computing each sample set as a direct extension of the polyhedral homotopy of Huber and Sturmfels. In particular, our goal is to connect *all* sample sets through solution paths defined by a single homotopy.

3.3. Square system formulation. In general, the toric slicing system (3.2) (in Definition 3.3) is a system of $q+d$ Laurent polynomials in n variables. While it is possible to study such systems directly, it is much more convenient to turn such system into square systems. In the following, let $r = n - q$. As noted in Subsection 2.4, without loss of generality, we only need to focus on cases where $n \geq q$, and hence $r \geq 0$. From Theorem 2.8, we can derive the following result.

Lemma 3.6. *If $d > r$, let $\Lambda = [\lambda_{i,j}]$ be a complex $n \times (d-r)$ matrix and consider the system of n Laurent polynomials in n variables*

$$F_{\square}^{(d)}(x_1, \dots, x_n) = F_{\square}^{(d)}(\mathbf{x}) = \begin{cases} \left(\mathbf{c}_i + \sum_{k=r+1}^{d} \lambda_{i,k}\mathbf{c}_k^*\right) \cdot \mathbf{x}^A & \text{for } i = 1, \dots, q \\ \left(\mathbf{c}_i^* + \sum_{k=r+1}^{d} \lambda_{q+i,k}\mathbf{c}_k^*\right) \cdot \mathbf{x}^A & \text{for } i = 1, \dots, r \end{cases}$$

For generic choices of Λ, all isolated points $\mathcal{V}^*(F^{(d)})$ are also isolated points in $\mathcal{V}^*(F_{\square}^{(d)})$. Furthermore, $\mathcal{V}^*(F^{(d)})$ and $\mathcal{V}^*(F_{\square}^{(d)})$ have the exact same set of positive dimensional irreducible components.

This transformation turns a toric slicing system into a square system while capturing all the $\mathbb{C}^*$-zeros. It is possible for this transformation to introduce extraneous zeros, i.e., isolated points that are in $\mathcal{V}^*(F_{\square}^{(d)}) \setminus \mathcal{V}^*(F^{(d)})$, but they can be filtered out easily. As we shall see, these extraneous zeros are far from useless. On the contrary, they are crucial in our construction of homotopy paths that will chain all sample sets together.

In the following, $\mathcal{V}^*(F_{\square}^{(d)})$ will be referred to as the **rank d sample superset** of F. Again, the points in these sets depend on the choices of $\{\mathbf{c}_k^*\}$, but the choices are of little interest as they are assumed to be generic.

Remark 3.7. In the special case of $n = q$, i.e., F being a square system, the corresponding system $F^{(d)}$ can be expressed concisely as

$$F_{\square}^{(d)}(\mathbf{x}) = (C + \Lambda\, C^*)(\mathbf{x}^A)^{\top}$$

where C, C^*, Λ are complex matrices of sizes $n \times m$, $d \times m$, and $n \times d$, respectively, given by

$$C = \begin{bmatrix} \mathbf{c}_1 \\ \vdots \\ \mathbf{c}_n \end{bmatrix}, \qquad C^* = \begin{bmatrix} \mathbf{c}_1^* \\ \vdots \\ \mathbf{c}_d^* \end{bmatrix}, \qquad \Lambda = \begin{bmatrix} \lambda_{11} & \cdots & \lambda_{1d} \\ \vdots & \ddots & \vdots \\ \lambda_{n1} & \cdots & \lambda_{nd} \end{bmatrix}.$$

In this form, it is easy to see that $F_\square^{(d)}(\mathbf{x})$ is exactly a perturbed version of the original system F in which the coefficient matrix C is replaced by $C + \Lambda C^*$ where ΛC^* is a generic matrix of rank d. This interpretation justifies the usage of the term "rank" in "rank d sample superset".

3.4. Stratified polyhedral homotopy. We now construct the homotopy method that can compute the sample supersets $\mathcal{V}_0^*(F_\square^{(d)})$, which contains the rank d sample sets of F, for each $d = 1, \dots, n$ using a single homotopy procedure. The first component in this procedure is the natural connection between consecutive sample supersets.

Definition 3.8. *Given the square system $F_\square^{(d)}$ defined above, we define $H^{(d)} : \mathbb{C}^n \times \mathbb{C} \to \mathbb{C}$, given by*

$$H^{(d)}(\mathbf{x}, t) = \begin{cases} \left(\mathbf{c}_i + \sum_{k=r+1}^{d-1} \lambda_{ik} \mathbf{c}_k^* + t\lambda_{i,d} \mathbf{c}_d^* \right) \cdot \mathbf{x}^A & \text{for } i = 1, \dots, q \\ \left(\mathbf{c}_i^* + \sum_{k=r+1}^{d-1} \lambda_{q+i,k} \mathbf{c}_k^* + t\lambda_{q+i,d} \mathbf{c}_d^* \right) \cdot \mathbf{x}^A & \text{for } i = 1, \dots, r \end{cases} \tag{3.3}$$

Clearly, $H^{(d)}(\mathbf{x}, 0) \equiv F^{(d-1)}(\mathbf{x})$ and $H^{(d)}(\mathbf{x}, 1) \equiv F^{(d)}(\mathbf{x})$. Furthermore, by restricting t to the real interval $[0, 1]$, we get a homotopy function between $F_\square^{(d-1)}$ and $F_\square^{(d)}$ since $H^{(d)}$ is continuous in both $\mathbf{x}$ and t. We shall show that the isolated $\mathbb{C}^*$-zeros of $H^{(d)}$ also move smoothly, as t goes from 1 to 0, forming smooth solution paths in $(\mathbb{C}^*)^n \times (0, 1]$.

Theorem 3.9. *For generic choices of $\mathbf{c}_k^*$'s and $\{\lambda_{i,j}\}$, the zero set of $H^{(d)}$ in $(\mathbb{C}^*)^n \times (0, 1]$ consists of finitely many smooth solution paths in $(\mathbb{C}^*)^n \times (0, 1]$ emanating from the nonsingular points of $\mathcal{V}_0^*(F_\square^{(d)})$ at $t = 1$, and the set of limit points of these paths in $(\mathbb{C}^*)^n$ as $t \to 0$ contains* all *nonsingular points in $\mathcal{V}_0^*(F_\square^{(d-1)})$.*

Proof. Define

$$G(\mathbf{x}, \mathbf{p}) = \begin{cases} \left(\mathbf{c}_i + \sum_{k=r+1}^{d-1} \lambda_{ik} \mathbf{c}_k^* + \lambda_{i,d}\, \mathbf{p} \right) \cdot \mathbf{x}^A & \text{for } i = 1, \dots, q \\ \left(\mathbf{c}_i^* + \sum_{k=r+1}^{d-1} \lambda_{q+i,k} \mathbf{c}_k^* + \lambda_{q+i,d}\, \mathbf{p} \right) \cdot \mathbf{x}^A & \text{for } i = 1, \dots, r \end{cases}$$

which represents a family of Laurent polynomials systems parametrized by $\mathbf{p} \in \mathbb{C}^m$ that contains $H^{(d)}(\mathbf{x}, t)$ for all t since $H^{(d)}(\mathbf{x}, t) = G(\mathbf{x}, t\, \mathbf{c}_d^*)$. By the Parameter Homotopy Theorem (Theorem 2.4), for generic choices of $\mathbf{p} \in \mathbb{C}^m$ the total number of nonsingular points in $\mathcal{V}_0^*(G(\mathbf{x}, \mathbf{p}))$, as a Laurent polynomial system in $\mathbf{x}$, is finite, and it is the same number, say $\mathcal{N}$. The exceptional set Q of the parameters for which the number of nonsingular points in $\mathcal{V}_0^*(G(\mathbf{x},\mathbf{p}))$ is less than $\mathcal{N}$ is contained in a proper algebraic set. In particular, at $t=1$, $H(\mathbf{x},t)=G(\mathbf{x},\mathbf{c}_d^*)$, so for generic choices of $\mathbf{c}_d^*$, the total number of starting points, i.e., the isolated points in $\mathcal{V}^*(H^{(d)})=\mathcal{V}^*(F_\square^{(d)})$ is exactly $\mathcal{N}$.

Our focus is therefore the path of evolution of this family between $\mathbf{p}=\mathbf{c}_d^*$ to $\mathbf{p}=\mathbf{0}$. In particular, this path can be parametrized as

$$\mathbf{p}(t)=t\mathbf{c}_d^*=(1-t)\mathbf{0}+t\mathbf{c}_d^*.$$

For almost all choices of $\mathbf{c}_d^*$, this path avoids the exceptional set Q in the parameter space [40, Lemma 7.1.2]. Following from the Parameter Homotopy Theorem (Theorem 2.4), as t goes from 1 to 0, the nonsingular isolated solutions to $H^{(d)}(\mathbf{x},t)=0$ form exactly $\mathcal{N}$ smooth paths (smoothly parametrized by t) emanating from the set of isolated points in $\mathcal{V}^*(F^{(d)})$ and reach all isolated points of $\mathcal{V}_0^*(F^{(d-1)})$ as limit points. ∎

The homotopy continuation procedure that tracks the solution paths defined by $H^{(d)}$ as t moves from 1 to 0 produces both the rank $d-1$ sample superset for F and the starting points for $H^{(d-1)}$. This chain reaction thus can continue until $H^{(r+1)}$ produces the rank r (the lowest rank) sample superset for F. This is the stratified polyhedral homotopy.

Definition 3.10 (Unmixed stratified polyhedral homotopy). *For an unmixed system F (3.1), in standard form, of q Laurent polynomials in n variables $\mathbf{x}=(x_1,\dots,x_n)$ and generic lifting function $\boldsymbol{\omega}:\operatorname{supp}(F)\to\mathbb{Q}^+$, we define $H:\mathbb{C}^n\times\mathbb{C}^{q+1}\to\mathbb{C}$ given by*

$$H(\mathbf{x},\mathbf{t})=\begin{cases}\left(\mathbf{c}_i+\sum_{k=r+1}^{n}t_{k-r}\lambda_{ik}\mathbf{c}_k^*\right)\cdot(\mathbf{x}^A\circ e^{-Mt_0\boldsymbol{\omega}}) & \text{for } i=1,\dots,q\\ \left(\mathbf{c}_i^*+\sum_{k=r+1}^{n}t_{k-r}\lambda_{q+i,k}\mathbf{c}_k^*\right)\cdot(\mathbf{x}^A\circ e^{-Mt_0\boldsymbol{\omega}}) & \text{for } i=1,\dots,r\end{cases} \tag{3.4}$$

where $\mathbf{t}=(t_0,t_1,\dots,t_q)$ and M is a sufficiently large positive real number.

Here, "$\circ$" denotes the entry-wise product between two row vectors of the same length, which is the group operation for $(\mathbb{C}^*)^m$. The constant $M\in\mathbb{R}^+$ is the same constant used in (2.1), which can be computed from the Newton polytope of H.

The starting points of the homotopy paths at $\mathbf{t}=(1,\dots,1)$ can be obtained by the same process that bootstraps the polyhedral homotopy (a brief review of this process is included in Appendix A, for completeness). Indeed, all $\mathbb{C}^*$-zeros of $H(\mathbf{x},(1,\dots,1))$ are isolated and nonsingular and the total number is exactly

$$n!\operatorname{vol}(\operatorname{conv}(\operatorname{supp}(F))),$$

which is also known as the *normalized volume* of the common Newton polytope $\operatorname{conv}(\operatorname{supp}(F))$. To obtain sample super set for F of ranks $n,n-1,\dots,n-q$, we could apply the standard homotopy continuation procedure on H along the piecewise linear parameter path

$$(1,\dots,1)\to(0,1,\dots,1)\to(0,0,1,\dots,1)\to\cdots(0,\dots,0,1)\to(0,\dots,0),$$

in the $\mathbf{t}$-space, starting from the initial points provided by the bootstrapping process of polyhedral homotopy. The parameter path consists of $q+1$ piecewise linear segment, and at the end of each segment, the projection of the solution paths onto the $\mathbf{x}$-coordinates generates the sample supersets for F of ranks $n, n-1, \dots, n-q$. Note that this homotopy can be formulated as a single homotopy function

(3.5)
$$H^*(\mathbf{x}, s) = \begin{cases} H(\mathbf{x}, (1-(1-s)(q+1), 1, 1, \dots, 1) & 1 \geq s > 1 - 1/(q+1)) \\ H(\mathbf{x}, (0, 2-(1-s)(q+1), 1, \dots, 1) & 1 - 1/(q+1) \geq s > 1 - 2/(q+1)) \\ \vdots & \vdots \\ H(\mathbf{x}, (0, \dots, 0, n+1-(1-s)(q+1)) & 1 - n/(q+1) \geq s > 0). \end{cases}$$

We summarize this algorithm in Algorithm 3.1.

Algorithm 3.1 Unmixed stratified polyhedral homotopy algorithm for regular zeros

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Require: An unmixed Laurent system F in standard form, lifting function $\boldsymbol{\omega} : S \to Q^+$ with generic images, and generic complex vectors $\mathbf{c}_1^*, \dots, \mathbf{c}_n^* \in \mathbb{C}^m$

Ensure: Returns finite sample sets $(W_n, \dots, W_0)$ such that, for $d = 1, \dots, n$, W_d intersects each d-dimensional reduced irreducible component of $\mathcal{V}^*(F)$.

1: Define $\tilde{X}_{-1} = \mathrm{PolyhedralBootstrap}(F, \boldsymbol{\omega})$
2: Define $\mathbf{t} = (t_0, t_1, \dots, t_q) = (1, 1, \dots, 1)$
3: **for** $k = 0, \dots, q$ **do**
4:    Define $X_k = \mathrm{HomotopyContinuation}(H, \tilde{X}_{k-1}, \mathbf{t};\ t_k : 1 \to 0)$
5:    Define $\tilde{X}_k = \{\mathbf{x} \in X_k \mid DF_{\square}^{(n-k)}(\mathbf{x})$ is nonsingular$\}$
6:    Define $W_{n-k} = \{\mathbf{x} \in \tilde{X}_k \mid F^{(n-k)}(\mathbf{x}) = \mathbf{0}$ and $\operatorname{rank} DF^{(n-k)}(\mathbf{x}) = n - k\}$
7:    Let $t_k = 0$
8: **end for**
9: **return** $(W_n, \dots, W_0)$

In this algorithm description, the subroutine PolyhedralBootstrap is responsible for bootstrapping the polyhedral homotopy method, as described in Subsection 2.1, for a given Laurent polynomial system and a generic lifting function. That is, it provides the isolated $\mathbb{C}^*$-solutions to the equation $H(\mathbf{x}, (1, \dots, 1)) = \mathbf{0}$. This process is reviewed in Appendix A. Subroutine HomotopyContinuation is the standard homotopy continuation method. In particular, HomotopyContinuation$(H, \tilde{X}_k, \mathbf{t}, t_k : 1 \to 0)$ tracks the paths defined by the equation $H = \mathbf{0}$ in $\mathbb{C}^n \times (0, 1]$ starting from the points in $\tilde{X}_k$ at $t_k = 1$ toward $t_k \to 0$. Other variables in $\mathbf{t} = (t_0, \dots, t_q)$ are held constant. The limit points within $(\mathbb{C}^*)^n$ are collected and returned as the result of this procedure.

3.5. Numerical considerations. In practice, homotopy continuation methods are generally implemented as numerical algorithms. Consequently, the sets $\tilde{X}_k$ in Algorithm 3.1 are only numerical approximations of the zeros in question, and therefore, the condition that $F^{(n-k)}(\mathbf{x}) = \mathbf{0}$, in Line 5, and the rank conditions in Lines 5 and 6 must be replaced by numerically well posed conditions.

For example, the condition $F^{(n-k)}(\mathbf{x}) = \mathbf{0}$ may be replaced by the numerically meaningful backward error condition that $F_\epsilon^{(n-k)}(\mathbf{x}) = \mathbf{0}$ for some threshold $\epsilon > 0$ and Laurent system $F_\epsilon^{(n-k)}$ with the same support such that $\|F_\epsilon^{(n-k)} - F^{(n-k)}\| < \epsilon$.

Similarly, the rank condition for the Jacobian matrices $DF_\square^{(n-k)}(\mathbf{x})$ and $DF^{(n-k)}(\mathbf{x})$ may be replaced by bounding on the ratio of the maximum and minimum singular values of $DF^{(n-k)}(\mathbf{x})$. A more robust and elegant solution is to frame these problems as well-studied *rank revealing problems* [7].

3.6. Combining steps. Algorithm 3.1 is presented to have the steps operating in serial along the piecewise linear parameter path. In practice, this arrangement is neither necessary nor efficient, since users generally have good a priori knowledge or educated guess about the maximum dimension of the zero sets. At very least, unless the system $F = (f_1, \dots, f_q)$ in n variables is trivial, the dimension of its $\mathbb{C}^*$-zero set must be strictly less than n. In this case, there is no need to directly compute the rank n sample superset, and it is sufficient to track the solution paths over the modified parameter path that starts with the line segment

$$(1, \dots, 1) \to (0, 0, 1, \dots, 1) \to \cdots$$

in Algorithm 3.1, i.e., the line segment given by $s \mapsto (s, s, 1, \dots, 1)$. Along this line segment in the parameter space, the polyhedral homotopy and the perturbation of coefficients are operating simultaneously, and at the end of this line segment, rank $n-1$ sample superset is produced.

In general, if it is known that the dimension of the $\mathbb{C}^*$-zero set of F is no more than $d_{\max}$, then it is sufficient to track the solution paths over the parameter path that starts with the line segment

$$(1, \dots, 1) \to (\underbrace{0, \dots, 0}_{d_{\max}+1}, 1, \dots, 1) \to \cdots$$

At the end of this first segment, rank $d_{\max}$ sample superset is produced which necessarily contain sample points for each reduced $d_{\max}$-dimensional irreducible components of $\mathcal{V}^*(F)$.

4. Reducing general cases to standard unmixed cases. The constructions presented so far requires the target Laurent system to be of a very special form — the "standard unmixed form" as defined in Definition 3.1. In this section, we describe how the general cases can be reduced to such special cases. As reviewed in Section 3, conditions 2 and 3 of Definition 3.1 can be satisfied by simple transformations, while condition 1 simply eliminates trivial cases for which much simpler methods can be used to solve them.

4.1. Lattice reduction for nonstandard unmixed systems. We now briefly outline the transformation required to satisfy the last condition (Condition 4) in Definition 3.1, i.e. the torsion-free condition.

Suppose the invariant factors of the support matrix A are $d_1, \dots, d_n \neq 0$. Let $P \in M_{n \times n}(\mathbb{Z})$ and $Q \in M_{m \times m}(\mathbb{Z})$ be the unimodular matrices in the Smith Normal Form

$$PAQ = \begin{bmatrix} D & \mathbf{0} \end{bmatrix} \qquad \text{where} \qquad D = \begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{bmatrix}$$

With these, we define matrices

$$L = P^{-1} D P \in M_{n\times n}(\mathbb{Z}) \qquad\qquad \tilde{A} = \begin{bmatrix} P^{-1} & \mathbf{0} \end{bmatrix} Q^{-1} \in M_{n\times m}(\mathbb{Z}). \tag{4.1}$$

Then $\tilde{A}$ also has full row rank, and we can verify that

$$P \tilde{A} Q = P \begin{bmatrix} P^{-1} & \mathbf{0} \end{bmatrix} Q^{-1} Q = \begin{bmatrix} I & \mathbf{0} \end{bmatrix}.$$

That is, systems with support matrix $\tilde{A}$ would satisfy the torsion-free condition (Condition 4 in Definition 3.1). We introduce the new variables $\mathbf{y} = (y_1, \ldots, y_n)$ via the relation

$$\mathbf{y} = \mathbf{x}^L \tag{4.2}$$

By Lemma 2.1, this defines a d-fold cover over $(\mathbb{C}^*)^n$, where $d = d_1 \cdots d_n = \det L$. That is, for each $\mathbf{y} \in (\mathbb{C}^*)^n$, there are precisely d distinct choices of $\mathbf{x} \in (\mathbb{C}^*)^n$ that would satisfy the above equation. With this change of variables

$$\mathbf{y}^{\tilde{A}} = (\mathbf{x}^L)^{\tilde{A}} = \mathbf{x}^{L\tilde{A}} = \mathbf{x}^{P^{-1}DP[P^{-1}\,\mathbf{0}]Q^{-1}} = \mathbf{x}^{P^{-1}[D\,\mathbf{0}]Q^{-1}} = \mathbf{x}^{A}$$

Therefore, via the change of variables (4.2), we can replace the original Laurent polynomial system F with support matrix A by a new system in $\mathbf{y}$ with support matrix $\tilde{A}$

$$\tilde{F}(\mathbf{y}) = \begin{cases} \mathbf{c}_1 \cdot \mathbf{y}^{\tilde{A}} \\ \quad\vdots \\ \mathbf{c}_1 \cdot \mathbf{y}^{\tilde{A}} \end{cases}$$

for which the stratified polyhedral homotopy defined in the previous section can be applied, and the $\mathbb{C}^*$-zero set $\mathcal{V}^*(F)$ is a d-fold cover over $\mathcal{V}^*(\tilde{F})$ defined by the map (4.2).

4.2. Turning mixed cases into unmixed cases. The description in Section 3 applies only to unmixed Laurent system, i.e., systems of Laurent polynomials with a common support. This constraint can be removed easily by considering generic linear combinations of the Laurent polynomials. We now consider a "mixed" Laurent system $F = (f_1, \ldots, f_q)$ in which the supports $\operatorname{supp}(f_1), \ldots, \operatorname{supp}(f_q)$ are not identical. With a generic complex nonsingular $q \times q$ matrix R, a mixed system $F = (f_1, \ldots, f_q)$ in $\mathbf{x} = (x_1, \ldots, x_n)$ can be turned into an equivalent randomized system

$$F^R(\mathbf{x}) = R\,F(\mathbf{x}).$$

Here, $F(\mathbf{x})$ is considered as a column vector. These two systems are equivalent in the sense that $\mathcal{V}^*(F) = \mathcal{V}^*(F^R)$. Yet, under the genericity assumption, there is no cancellation of the terms in RF, and hence F^R is unmixed. The stratified polyhedral homotopy construction described in Section 3 can therefore be applied to the unmixed system F^R instead.

Since the support of RF is $S_1 \cup \cdots \cup S_q$, where $S_i = \operatorname{supp}(f_i)$ for $i = 1, \ldots, q$, the number of paths defined by the stratified polyhedral homotopy, i.e. the BKK bound of F^R, is

$$n! \operatorname{vol}(\operatorname{conv}(S_1 \cup \cdots \cup S_q)). \tag{4.3}$$

In the rest of this paper, this bound will be referred to as the *Kushnirenko bound* to emphasize the fact that the unmixed version of the BKK bound is used.

In summary, the framework developed here can also be applied to mixed Laurent systems simply by considering random linear combinations of the Laurent polynomials in the system. We conclude this section with a few remarks on the more subtle points.

Remark 4.1. In the case of $q = n$, i.e. F being a square system, it is well known that

$$n! \operatorname{vol}(\operatorname{conv}(S_1 \cup \cdots \cup S_n)) \geq \operatorname{mvol}(\operatorname{conv}(S_1), \ldots, \operatorname{conv}(S_n)). \tag{4.4}$$

This follows from the monotonicity of the mixed volume function. That is, the transformation $F \mapsto RF$ may or may not increase the BKK bound, which is the number of homotopy paths defined by the stratified polyhedral homotopy. Conditions for the equality of the two was first discovered by Maurice Rojas in 1994 [31]. Variations of these conditions have since been rediscovered a couple of times [6, 8]. As listed in Ref. [8], for many important families of Laurent systems derived from applied sciences, the two sides of (4.4) are identical, and thus the randomization process does not inflate the number of homotopy paths one has to track using the unmixed version of the stratified polyhedral homotopy method.

Remark 4.2. It should be noted that the transformation $F \mapsto RF$ is not invariant under lattice translations of the supports, even though the $\mathbb{C}^*$-zero set they define is: For the Laurent system $F = (f_1, \ldots, f_q)$ and any set of Laurent monomials $\mathbf{x}^{\mathbf{v}_1}, \ldots, \mathbf{x}^{\mathbf{v}_q}$, with $\mathbf{v}_1, \ldots, \mathbf{v}_q \in \mathbb{Z}^n$, the Laurent system $(\mathbf{x}^{\mathbf{v}_1} f_q, \ldots, \mathbf{x}^{\mathbf{v}_q} f_q)$ also has the exact same $\mathbb{C}^*$-zero set. Yet, the randomized system $R(\mathbf{x}^{\mathbf{v}_1} f_q(\mathbf{x}), \ldots, \mathbf{x}^{\mathbf{v}_q} f_q(\mathbf{x}))^\top$ can be quite different from $F^R = RF$. In particular, the Kushnirenko bound (4.3), i.e. the number of paths the stratified polyhedral homotopy will define, may be different depending on the choices of $\mathbf{v}_1, \ldots, \mathbf{v}_q$. Finding the optimal choice so that $n! \operatorname{vol}(\operatorname{conv}(S_1 + \mathbf{v}_1 \cup \cdots \cup S_q + \mathbf{v}_q))$ is minimized is still an open problem.

5. Decomposition of the BKK bound.

Bernshtein's first theorem (Theorem 2.3) states that for a system of n Laurent polynomials $(f_1, \ldots, f_n)$ in n variables, the number of isolated zeros in $(\mathbb{C}^*)^n$ is bounded by the mixed volume $\operatorname{mvol}_n(P_1, \ldots, P_n)$, where $P_1, \ldots, P_n$ are the Newton polytopes of $f_1, \ldots, f_n$, respectively. It equals the normalized volume $n! \operatorname{vol}(P)$ in the unmixed case, i.e., when $P_1, \ldots, P_n = P$ (Theorem 2.2). This is the BKK bound. Indeed, for generic coefficients, all $\mathbb{C}^*$-zeros are isolated, and this bound is exact. When positive-dimensional components are present, however, the number of isolated $\mathbb{C}^*$-zeros will be strictly less than this bound. A natural question to ask in this situation is whether it is possible to decompose the BKK bound as a sum of local contributions from each irreducible component

This question mirrors the classical question of how to decompose the Bézout number. As early as 1680, Newton already observed that the number of isolated intersections between two planar curves of degrees d_1, d_2 is bounded by $d_1 \cdot d_2$. In 1764, Bézout proved this upper bound can be reached if the curves are in general positions, and the same bound applies to the isolated zeros of a system of n polynomials in $\mathbb{CP}^n$. This is the Bézout bound. Indeed, when there are no positive-dimensional components and intersections are counted with multiplicities, this bound can always be reached with equality. When positive-dimensional components are present, however, the naive interpretation of this bound breaks down. The search for a decomposition of the Bézout bound into local contributions from all components of a polynomial system thus began.

Among the great variety of different (but ultimately equivalent) approaches in constructing such a decomposition of the Bézout bound, the dynamic approach proposed by Severi [34] and subsequently corrected by Lazarsfeld [20] is the most relevant here. By assigning an integer multiplicity to each subvariety of the projective zero set of a polynomial system, they established such a decomposition of the Bézout bound.

The stratified polyhedral homotopy method described above produces a similar assignment of multiplicity as a by-product, at least for unmixed cases involving reduced components. First, through a routine application of the Parameter Homotopy Theorem (Theorem 2.4), we can verify that even though the points in the sample sets $W_n, \ldots, W_0$, produced by Algorithm 3.1, depends on the random choices of the coefficients $\mathbf{c}_1^*, \ldots, \mathbf{c}_n^*$, the number of points within each sample set remains a constant.

Proposition 5.1. *If all components of $\mathcal{V}^*(F)$ of dimension d are (generically) reduced, then for generic choices of $\mathbf{c}_1^*, \ldots, \mathbf{c}_n^* \in \mathbb{C}^m$, the number of distinct points in the rank-d sample set W_d is a constant that is independent of the choices of $\mathbf{c}_1^*, \ldots, \mathbf{c}_n^*$.*

Since each point in a sample set W_{d_i} is produced by a homotopy path, and the total number of homotopy paths is the Kushnirenko bound (4.3), by counting the points in each W_{d_i}, we have a crude extension of the this bound that take into considerations of the contributions from components of each dimension.

Proposition 5.2. *Suppose the $\mathbb{C}^*$-zero set of a Laurent system $(f_1, \ldots, f_n)$ consists of components $C_{d_1}, \ldots, C_{d_\ell} \neq \varnothing$ where each C_{d_i} is the union of all d_i-dimensional components. Let $S = \operatorname{supp}(f_1) \cup \cdots \cup \operatorname{supp}(f_n)$ and $\jmath(C_{d_i}) = |W_{d_i}|$, then $\jmath(C_{d_i}) > 0$ and*

$$\sum_{i=1}^{\ell} \jmath(C_{d_i}) \leq n! \operatorname{vol}_n(\operatorname{conv}(S)). \tag{5.1}$$

This bound can be refined significantly. By extending the function $\jmath$ to individual irreducible components in each C_{d_i} via restriction (see the remark in Subsection 7.1 for the connection to the stronger irreducible decomposition), we have a more refined decomposition of the Kushnirenko bound in terms of contributions from irreducible components.

In addition, by broadening the concept of sample points and components in the above proposition, we can reach an exact decomposition of the Kushnirenko bound in certain cases. First, we can take into consideration end points of homotopy paths that are filtered out by the rank condition in Line 5 of Algorithm 3.1 (singular sample points) as well as divergent paths (sample points at toric infinity), and count them with proper multiplicity. Second, we need to include subvarieties of $\mathcal{V}^*(f_1, \ldots, f_n)$ that may or may not be irreducible components into the left-hand side of (5.1), as long as they attract homotopy paths defined by Algorithm 3.1. In other words, we need to include "distinguished" subvarieties as constructed in Ref. [14]. The full development of this theoretical aspect is outside the scope of this paper, which focuses on the numerical aspect of this problem. We will, instead, illustrate the exact decomposition of the BKK bound through a few concrete examples in Section 6 (e.g., equation (6.2)).

6. Examples.

In this section, we present results from numerical experiments in applying the proposed algorithm to compute sample points of positive dimensional $\mathbb{C}^*$-solution sets of some well known Laurent polynomial systems.

All experiments are carried out with a proof-of-concept implementation that uses libDH [9] as the path tracker which utilizes GPU acceleration. For a system in n unknowns, we use the stratified polyhedral homotopy of type-$(n-1, n-2, \ldots, 1, 0)$ to compute sample sets (which ignores the possibility of n-dimensional components).

Internally, calculations, with few exceptions that will be noted below, are carried out strictly in double-precision floating point numbers, in order to test the robustness of the proposed numerical algorithm. Therefore, in the following, words such as "on", "in", and "reach" should be interpreted as points or homotopy paths being sufficiently close to points or positive-dimensional components up to a tolerance appropriate for double-precision floating point calculations. Since the goal is to verify the expected behavior against known solution sets, no certification of the solutions are performed.

6.1. The running example. In the running example (Example 1.1) we considered the polynomial system

$$F(x_1, x_2) = \begin{cases} (x_1^2 + x_2^2 - 9)(x_1 + x_2 - 3) \\ (x_1^2 + x_2^2 - 9)(x_1 - x_2 - 1). \end{cases}$$

Its $\mathbb{C}^*$-zero set $\mathcal{V}^*(F)$ consists of the 1-dimensional irreducible component $Q = \mathcal{V}^*(x_1^2 + x_2^2 - 9)$ and the isolated and nonsingular point $P = (x_1, x_2) = (2, 1) \notin Q$. Both components are (generically) reduced.

It is easy to verify that the convex hull of the union of the supports is the simplex defined by $\{(0,0), (3,0), (0,3)\}$, which has normalized volume of 9. That is, its Kushnirenko bound is 9. Therefore, the stratified polyhedral homotopy defines 9 homotopy paths.

- At the end of the first stage of the homotopy, 6 paths reach 6 (nonsingular) rank-1 sample points (each reached exactly once) inside the 1-dimensional component Q.
- The remaining 3 paths continue onto the second stage, and one of them reaches one (nonsingular) rank-0 sample point, which coincide with the only isolated zero $P = (2, 1)$. The remaining two paths converge to points in Q or its projective closure.

This shows that by following the homotopy paths defined by a single homotopy, both sample points of 1-dimensional component and the isolated zero can be reached.

6.2. Algebraic Kuramoto equations on homogeneous networks. The Kuramoto model emerged from the study of networks of oscillators, which can be modeled as collections of points on the complex plane circling 0 while pulling on one another. They have found many real-world applications. Kuramoto proposed a simple yet illuminating dynamical system governed

$$\dot{\theta}_i = \omega_i - \sum_{j \sim i} k_{ij} \sin(\theta_i - \theta_j) \quad \text{for } i = 0, 1, \ldots, N-1. \tag{6.1}$$

Here, N is the number of oscillators, which are labeled as $i = 0, 1, \ldots, N-1$. θ_i is the phase angle of the i-th oscillator, which describe its state, and ω_i is its natural frequency (relative to the mean frequency). $i \sim j$ indicates oscillators i and j are coupled, in which case the coupling coefficient $k_{ij} = k_{ji}$ quantifies how strongly they influence one another. Due to the inherent rotational invariance, we can fix $\theta_0 = 0$, and discard the equation for $i = 0$.

Fundamental to the study of this model is the problem of finding frequency synchronization configurations, which are configurations $(\theta_1, \ldots, \theta_{N-1})$ of the network for which $\dot{\theta}_i = 0$ for all

$i = 1, \ldots, N-1$, i.e., the equilibria of (6.1). Though the equilibrium equation for (6.1) is not algebraic, with the change of variables $x_i = e^{\mathrm{i}\theta_i}$, the synchronization configurations can be described by the system of Laurent polynomial equations

$$0 = \omega_i - \sum_{i \sim j} \frac{k_{ij}}{2\mathrm{i}} \left(x_i x_j^{-1} - x_j x_i^{-1} \right) \quad \text{for } i = 1, \ldots, N-1.$$

This is the algebraic Kuramoto equation. Its Bézout number and bi-homogeneous Bézout number are $2^{2(N-1)}$ and $\binom{2(N-1)}{N-1}$, respectively [2]. Its Kushnirenko bound and BKK bound are identical, and it can be much lower than the Bézout numbers for sparse networks.

The network is said to be *homogeneous* if $\omega_i = 0$ for all $i = 0, \ldots, N-1$. This is the special case we shall consider here, since it was shown in Refs. [28] that under the homogeneity assumption, for specific choices of the coupling coefficients $\{k_{ij}\}$, there can be positive-dimensional solution sets.

6.3. The 4-cycle network. For a homogeneous network of 4 oscillators that form a 4-cycle, the corresponding algebraic Kuramoto system is given by

$$F_{C_4} = \begin{cases} -(x_1/x_0 - x_0/x_1) - (x_1/x_2 - x_2/x_1) \\ -(x_2/x_1 - x_1/x_2) - (x_2/x_3 - x_3/x_2) \\ -(x_3/x_2 - x_2/x_3) - (x_3/x_0 - x_0/x_3)\,, \end{cases}$$

where $x_0 = 1$ is the constant that corresponds to the reference phase of the system. The $\mathbb{C}^*$-zero set $V = \mathcal{V}^*(F_{C_4})$ contains two nonsingular isolated zeros $V_0 = \{(1,1,1), (-1,1,-1)\}$. There are also three 1-dimensional components parametrized by the monomial maps

$$\mathbf{x}_1(\lambda) = (-2\mathrm{i}\lambda, -1, 2\mathrm{i}\lambda)\,, \qquad \mathbf{x}_2(\lambda) = \left(2\mathrm{i}\lambda, -1, \frac{1}{2\mathrm{i}\lambda}\right), \qquad \mathbf{x}_3(\lambda) = \left(1/2\mathrm{i}\lambda, \frac{-1}{4\lambda^2}, \frac{-1}{2\mathrm{i}\lambda}\right),$$

respectively. In addition, there are two embedded points $E_1 = (-i,-1,i), E_2 = (i,-1,i)$ inside the 1-dimensional components. Indeed, they are the intersections of $V_{1,1}, V_{1,2}, V_{1,3}$. The existence of positive-dimensional components was discovered by Lindberg, Zachariah, Boston and Lesieutre. Detailed analysis of the solutions, including their stability properties, was provided by Sclosa [33]. Here, we utilize these existing knowledge to verify the expected behavior of the stratified polyhedral homotopy method.

The Kushnirenko bound of this system is 12, which is identical to its BKK bound [10, 8]. Therefore, the stratified polyhedral homotopy defines 12 homotopy paths.

Remark 6.1. It is worth noting the significant advantage of the proposed stratified polyhedral homotopy method over homotopy methods whose complexity is linear in Bézout bounds. The Bézout number of this system is $2^6 = 64$, while the bi-homogeneous Bézout number is $\binom{6}{3} = 20$. The BKK bound is only 12. Indeed, as noted in Ref. [10], the ratio between either Bézout number and the BKK bound goes to ∞ as $N \to \infty$.

- At the end of the first stage of the homotopy, no rank-2 sample points are produced, which signifies that there are no 2-dimensional components in the $\mathbb{C}^*$-zero set of this system. All 12 paths thus continue to the next stage.

- At the end of the second stage, 6 of the 12 paths reach 6 (nonsingular) rank-1 sample points inside the 1-dimensional components, two sample points on each of the component $V_{1,1}, V_{1,2}, V_{1,3}$. The remaining 6 paths continue to the next stage.
- At the end of the third stage, 2 of the 6 remaining paths converge to the two nonsingular isolated zeros $(1, 1, 1)$ and $(-1, 1, -1)$, respectively. The rest of the paths converge to two of the embedded points E_1 and E_2 (each reached twice).

In this case, there are no divergent paths (i.e., no paths escape $(\mathbb{C}^*)^3$), and thus, by counting the number of paths reaching each component, include the two embedded points, we have a full decomposition of the BKK bound into the local contributions from 7 components:

$$\text{mvol}(\mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3) = \text{vol}_3(\text{conv}(\mathcal{N}_1 \cup \mathcal{N}_2 \cup \mathcal{N}_3)) = 12 = \underbrace{2+2+2}_{\substack{\text{1-dimensional}\\ \text{components}}} + \underbrace{1+1}_{\substack{\text{Isolated}\\ \text{points}}} + \underbrace{2+2}_{\substack{\text{Embedded}\\ \text{points}}}, \tag{6.2}$$

where $\mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3$ are the Newton polytopes of three equations, respectively. This shows that the bound given in (5.1) may become an equality when all "distinguished" components are taken into consideration, thereby provides an exact decomposition of the BKK bound.

6.4. The 6-cycle network. Similar to the formulation above, the algebraic Kuramoto system for the 6-cycle graph contains 5 equations in 5 complex variables. It is shown in Ref. [11] that by picking coupling coefficients $k_{ij} = \pm s$ for some $s \in \mathbb{C}^*$ with an odd number of negative choices, the resulting Laurent system has 10 different 1-dimensional components, each having a monomial parametrization similar to those given above. Here, we choose $k_{ij} = 1$ for $\{i, j\} \neq \{0, 1\}$ and $k_{01} = k_{10} = -1$. The corresponding Laurent system is

$$F(x_1, x_2, x_3, x_4, x_5) = \begin{cases} +(x_1/x_0 - x_0/x_1) - (x_1/x_2 - x_2/x_1) \\ -(x_2/x_1 - x_1/x_2) - (x_2/x_3 - x_3/x_2) \\ -(x_3/x_2 - x_2/x_3) - (x_3/x_4 - x_4/x_3) \\ -(x_4/x_3 - x_3/x_4) - (x_4/x_5 - x_5/x_4) \\ -(x_5/x_4 - x_4/x_5) - (x_5/x_0 - x_0/x_5), \end{cases}$$

where $x_0 = 1$ is the constant corresponds to the reference phase as before. The Kushnirenko bound of this system is $6 \cdot \binom{6-1}{\lfloor (6-1)/2 \rfloor} = 60$. Therefore, the stratified polyhedral homotopy method defines 60 paths (in contrast with the Bézout number of 1024 or the bi-homogenous Bézout number of 252).

- No rank-d sample points are produced for all $d > 1$.
- 20 paths reach 20 (nonsingular) rank-1 sample points on the 1-dimensional components with two sample points on each component.
- The remaining 40 homotopy paths continue on and they reach isolated zeros of F as well as embedded points.

Together, these results provide numerical verifications of the results developed in Ref. [11]. Indeed, they provide strong numerical evidence suggesting that the positive-dimensional components described in [11, Proposition 5.2] are the only positive-dimensional components.

6.5. Nested distinguished components. In Ref. [3], the polynomial system

$$F(x,y,z) = \begin{cases} (xy-z)(x-y)(x+y-z) \\ (xy-z)(xy-z+(x-y)(x+2y-3z)) \\ (xy-z)(xy-z+(x-y)(2x-3y+z)) \end{cases}$$

is used as an example. The $\mathbb{C}^*$-zero set of F consists of a quadratic surface $Q = \mathcal{V}^*(xy-z)$ and the isolated point $P = (2/11, 10/11, 12/11) \notin Q$. There is also a distinguished 1-dimensional component $C = \mathcal{V}^*(x-y, xy-z)$ that is contained in Q. Let S_1, S_2, S_3 be the supports of the three Laurent polynomials in this system, then the Kushnirenko bound is $\mathrm{vol}_3(\mathrm{conv}(S_1 \cup S_2 \cup S_3)) = 12$. Therefore, the stratified polyhedral homotopy method defines 12 paths.

- At the end of the first stage, 11 paths converge to points in Q. However, not all of them produce nonsingular rank-2 sample points. Among them, two pairs of paths converge to two points in C (each reached twice).
- 1 path continue on and converge to P.

The important observation is that the existence of such a nested distinguished component does not prevent the stratified polyhedral homotopy from reaching nonsingular sample points for the 2-dimensional component and the isolated zero. Indeed, such a 1-dimensional distinguished component contained inside a 2-dimensional distinguished component can still be sampled, if we take into consideration the singular sample points.

6.6. Cyclic-4 system. The "Cyclic-n" family of polynomial systems have been used as standard test cases relating to solving polynomial systems. Among this family, the "Cyclic-4" system is the smallest system that has a positive-dimensional zero set. It is given by

$$F(x_1,x_2,x_3,x_4) = \begin{cases} x_1+x_2+x_3+x_4 \\ x_1x_2+x_2x_3+x_3x_4+x_4x_1 \\ x_1x_2x_3+x_2x_3x_4+x_3x_4x_1+x_4x_1x_2 \\ x_1x_2x_3x_4-1. \end{cases}$$

Its $\mathbb{C}^*$-zero set consists of two one-dimensional components as well as 8 embedded points. The Kushnirenko bound of this system is 22. Therefore, the stratified polyhedral homotopy method defines 22 paths.

- No (nonsingular) rank-d are produced for $d > 1$. This agrees with the fact that there are no components in $\mathcal{V}^*(F)$ of dimension greater than 1.
- At least 4 (nonsingular) rank-1 sample points are produced, two on each of the 1-dimensional components. In addition, 2 paths reach end points that are numerically singular (the condition number of $DF^1_\square$ exceeds 10^6).
- No (nonsingular) rank-0 sample point is produced. But 16 paths reach the 8 singular embedded points of $\mathcal{V}^*(F)$. Each is reached twice.

This example gives a clear illustration of the strength of the stratified polyhedral homotopy over the original polyhedral homotopy. When the original polyhedral homotopy is applied directly to solve this system, only the 8 embedded points are reached, which are singular zeros of $\mathcal{V}^*(F)$. In contrast, the stratified polyhedral homotopy method produces numerically nonsingular sample points on each of the 1-dimensional components, which can be used as input for higher level algorithms (e.g., irreducible decomposition, as noted in Subsection 7.1).

7. Concluding remarks.

The proposed stratified polyhedral homotopy method computes a special type of sample points for all reduced irreducible components of the $\mathbb{C}^*$-zero sets of a Laurent polynomial system. More specifically, when applied to a Laurent polynomial system F in n complex variables, the proposed homotopy defines a finite number of piecewise smooth homotopy paths in $(\mathbb{C}^*)^n$ (or a suitable compactification of it) that pass through finite sample sets $W_n, W_{n-1}, \ldots, W_1, W_0$ (which may be empty) such that W_d contains at least one point from each d-dimensional reduced irreducible component of the $\mathbb{C}^*$-zero set of F. Moreover, such sample points are smooth points in the sense that the nullity of the Jacobian matrix of F at these sample points match the local dimensions of the components there. This smoothness property is important, as it enables these sample points to generate additional information about the $\mathbb{C}^*$-zero set of F through higher level algorithms in numerical algebraic geometry. We conclude with a few remarks on these higher level algorithms that can use sample points produced by the proposed stratified polyhedral homotopy as input.

7.1. From sample sets to irreducible decomposition.

At each iteration of Line 6 of Algorithm 3.1, a finite set of points W_d is produced. Collectively, they form a numerically well-behaving representations of the d-dimensional components V_d of the $\mathbb{C}^*$-zero set $\mathcal{V}^*(F)$ of F. Therefore, the production of the sample sets $W_n, W_{n-1}, \ldots, W_1, W_0$ is a numerical equivalence of decomposing $\mathcal{V}^*(F)$ according to the dimensions of its components. A more refined decomposition is the *irreducible decomposition*. In particular, the d-dimensional component V_d may be further decomposed into its irreducible components

$$V_d = V_{d,1} \cup V_{d,2} \cup \cdots \cup V_{d,m_d}.$$

Under the assumption that these components are reduced, the numerical equivalence of this decomposition will be a partition of the rank d sample set W_d

$$W_d = W_{d,1} \cup W_{d,2} \cup \cdots \cup W_{d,m_d}$$

such that $W_{d,i} \subset V_{d_i}$ for each $i = 1, \ldots, m_d$. In principle, this partition may be produced through a *monodromy* algorithm [36]. The effectiveness and efficiency of such an approach will be important questions for future studies.

7.2. Sampling nonreduced components.

Our discussions focused only on (generically) reduced components. In general, the $\mathbb{C}^*$-zero set of a Laurent system F, may contain nonreduced components. That is, over a component V of the zero set, it is possible for the Jacobian matrix DF to have a nullity that is strictly greater than the dimension of a component V at every point. Such nonreduced component may result in isolated but singular end points in the set X_k in Line 4 of Algorithm 3.1. These points are filtered out in Line 5. Consequently, the proposed algorithm simply ignores the existence of nonreduced components.

The main reason for ignoring such nonreduced component is that singular end points in Line 4 of Algorithm 3.1 (i.e., points in $X_k \setminus \tilde{X}_k$) may become start points of "singular" homotopy paths in the homotopy continuation step in Line 4 for which *basic* path tracking algorithm cannot be applied.

While it is possible to applied more advanced algorithms to tracking such "singular" homotopy paths [37] and potentially reach singular sample points that serve as numerical representations of certain nonreduced components, within the numerical algebraic geometry community, however, it is much preferred to replace the equations that define the same zero set so that the nonreduced structure on the zero set disappears. These are special form of regularization processes. The most commonly used is a family of closely related symbolic preprocessing step collectively known as *deflation* [13, 22]. Combining the algorithm proposed here with deflation steps will be a natural extension that should be investigated.

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Appendix A. Bootstrapping unmixed polyhedral homotopy. For completeness, we briefly outline, without proofs, the main procedure for computing the starting solutions for the homotopy (3.4) (Definition 3.10), which are the nonsingular isolated zeros of $H(\mathbf{x}, (0, \ldots, 0))$. Without loss of generality, it is sufficient to assume F is an unmixed square system, and its support is in standard form (as defined in Definition 3.1). Under the genericity assumption for ω, the regular subdivision of S induced by the lifting function ω is a triangulation. That is, the projection of the lower hull of the lifted point configuration $\hat{S} = \{(\mathbf{a}, \omega(\mathbf{a})) \mid \mathbf{a} \in S\} \subset \mathbb{Q}^{n+1}$ form a triangulation for S. Let

$$T = \{\boldsymbol{\alpha} \in \mathbb{Q}^n \mid (\boldsymbol{\alpha}, 1) \text{ is an inner normal of a facet of } \hat{S}\}.$$

Then for each $\boldsymbol{\alpha} \in T$, the minimum of the linear functional $\langle \bullet, (\boldsymbol{\alpha}, 1) \rangle$ is achieved at exactly $n + 1$ points in $\hat{S}$. Let $\Delta(\boldsymbol{\alpha})$ be the projection of this subset of $n + 1$ points in S. Since the columns in the support matrix A and the coefficient matrix C (as used in Remark 3.7) correspond to points in S, we shall use the notations $A_{\Delta(\boldsymbol{\alpha})}$ and $C_{\Delta(\boldsymbol{\alpha})}$ for the submatrice of A and C, respectively, consisting of columns corresponding to the subset $\Delta(\boldsymbol{\alpha}) \subset S$. With these, we define

$$F^{(\boldsymbol{\alpha})}(\mathbf{x}) = C_{\Delta(\boldsymbol{\alpha})} \, (\mathbf{x}^{A_{\Delta}(\boldsymbol{\alpha})})^{\top}, \tag{A.1}$$

which is a square system of n Laurent polynomials each having exactly $n + 1$ terms. In Ref. [23], A. Leykin, J. Verschelde, and Y. Zhuang named such a system a "*simplex system*", since its Newton polytope is a simplex. The numerical issues involved in solving such a system is analyzed in the same article, and more detail is included in the Ph.D. thesis of

Y. Zhuang [43]. Through a toric transformation induced by the vector $\boldsymbol{\alpha}$, the solution to such a simplex system can be used as numerical approximations for the starting points of the homotopy paths for Algorithm 3.1.