Theoretical aspects of mixed volume computation via mixed subdivision

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We analyze the computation of mixed volume of tuples of polytopes via fine mixed subdivisions. This method expresses the mixed volume as a sum of easily computable standard volumes of polytopes called *mixed cells*. Mixed cells play a critically important role in *polyhedral homotopy* continuation methods, which in turn are a particularly efficient numerical method for the solution of systems of polynomial equations. We provide a complete and self-contained account of the underlying computational convexity techniques, assuming no background in algebraic geometry.

1. Introduction

Throughout this article, n represents an arbitrary but fixed positive integer. For a set $A \subset \mathbb{R}^m$, we denote its closure, interior, boundary and convex hull by $\bar{A}$, int A, ∂A and conv A respectively. For two sets $A, B \subset \mathbb{R}^m$, $A + B$ denotes their *Minkowski sum* [25] $\{\boldsymbol{a} + \boldsymbol{b} : \boldsymbol{a} \in A$ and $\boldsymbol{b} \in B\} \subset \mathbb{R}^m$. With a positive real number λ and a set $A \subset \mathbb{R}^m$, the scaling of A by the factor λ is simply the result of applying the scaling to each point of A, that is, $\lambda A = \{\lambda \boldsymbol{a} : \boldsymbol{a} \in A\}$. Furthermore, $\mathrm{Vol}_n A$ is the standard volume of $\mathbb{R}^n$, and $\langle \cdot, \cdot \rangle$ stands for the standard Euclidean inner product on $\mathbb{R}^n$.

For $i = 1, \ldots, n$, let $\mathcal{S}_i$ be a finite subset of $\mathbb{R}^n$ and $\mathcal{Q}_i = \mathrm{conv}\, \mathcal{S}_i$. Our interest will be focused on the n-tuple of sets $\mathcal{S} = (\mathcal{S}_1, \ldots, \mathcal{S}_n)$ and the n-tuple of convex polytopes $\mathcal{Q} = (\mathcal{Q}_1, \ldots, \mathcal{Q}_n)$. For positive real numbers $\lambda_1, \ldots, \lambda_n$, the n-dimensional volume of the Minkowski sum

$$\lambda_1 \mathcal{Q}_1 + \cdots + \lambda_n \mathcal{Q}_n = \{\lambda_1 \boldsymbol{q}_1 + \cdots + \lambda_n \boldsymbol{q}_n : \boldsymbol{q}_i \in \mathcal{Q}_i \text{ for } i = 1, \ldots, n\}$$

is a homogeneous polynomial of degree n in the variables $\lambda_1, \ldots, \lambda_n$. We call the coefficient of the monomial $\lambda_1 \times \cdots \times \lambda_n$ in this polynomial the **mixed volume** of $(\mathcal{Q}_1, \ldots, \mathcal{Q}_n)$ [10, 13, 25] , denoted by $\mathcal{M}(\mathcal{Q}_1, \ldots, \mathcal{Q}_n)$,

or simply $\mathcal{M}(\mathcal{Q})$. The mixed volume of convex polytopes is a classical concept in combinatorial geometry, dating back to works of Minkowski in the early 1900s. Mixed volume, thanks to its connections to Hodge theory and toric varieties (see, e.g., [6, 18, 28]), is an important invariant from convex geometry that has since proved important in numerous applications related to solving polynomial systems.

Among various approaches to computing the mixed volume $\mathcal{M}(\mathcal{Q})$ (such as [3, 7–9, 24]), we will detail *fine mixed subdivisions* of the n-tuple $\mathcal{S} = (\mathcal{S}_1, \ldots, \mathcal{S}_n)$ introduced in [16]. This method produces as a by-product the *mixed cells* which play a critically important role in the construction of efficient *polyhedral homotopy* continuation methods [5, 15, 16, 20, 21, 29] in solving polynomial systems numerically. Moreover, fine mixed subdivisions are also constructions of great interest in the context of *tropical algebraic geometry* [17] because they help represent intersections of tropical hypersurfaces [14].

This article is organized as follows: In §2, the concept of *fine mixed subdivisions* is introduced, followed by an exploration of a very important property of the fine mixed subdivision, its *scaling invariance*, in §3. We then detail the concrete method for computing mixed volume via a (fine) mixed subdivision in §4. In §5 a method for finding a (fine) mixed subdivision via *generic lifting* is discussed which also constitutes a constructive proof to the existence of a (fine) mixed subdivision. This method reduces the problem of mixed volume computation into a computation to list all mixed cells in a mixed subdivision. Special cases where some of polytopes in $\mathcal{Q} = (\mathcal{Q}_1, \ldots, \mathcal{Q}_n)$ are identical, known as *semi-mixed* cases, are studied in §6. It indicates that the computation procedure for computing mixed volume can be modified to take advantage of these special structures. We conclude the article in §7 with a brief discussion of the concrete formulation of the mixed cell enumeration problem in terms of *linear programming* problems.

2. Fine mixed subdivisions

Recall that a *convex polytope* $P \subset \mathbb{R}^n$ is the convex hull of finite many points in $\mathbb{R}^n$. Since all polytopes considered in this article are convex, they will therefore simply be known as polytopes. A subset F of polytope P is called a *face* of P if there exists $\boldsymbol{\alpha} \in \mathbb{R}^n$ for which the linear functional $f(\boldsymbol{x}) = \langle \boldsymbol{\alpha}, \boldsymbol{x} \rangle$ on $\mathbb{R}^n$ attains its minimum over P at F. In this case, the vector $\boldsymbol{\alpha}$ is called an *inner normal* of F. When P is an n-dimensional polytope in $\mathbb{R}^n$, its $(n-1)$-dimensional faces are called *facets* of P. A subdivision of a polytope $\widetilde{\mathcal{Q}}$ is a collection of polytopes intersecting only along their common faces

and whose union is the entire $\widetilde{\mathcal{Q}}$. As a point of departure, we generalize this concept to the context of an n-tuple of polytopes.

Definition 1 (Subdivision). A **subdivision** of $\mathcal{S} = (\mathcal{S}_1, \ldots, \mathcal{S}_n)$ is a collection $\mathcal{D}$ of n-tuples of the form $C = (C_1, \ldots, C_n)$, called **cells**, with $\varnothing \neq C_i \subseteq \mathcal{S}_i$ for each $i = 1, \ldots, n$ such that

(a): $\dim(\operatorname{conv} C_1 + \cdots + \operatorname{conv} C_n) = n$ for all $C \in \mathcal{D}$,

(b): For a distinct pair $A, B \in \mathcal{D}$, if $(\operatorname{conv} A) \cap (\operatorname{conv} B)$ is nonempty, then it is a common face of both,

(c): $\bigcup_{(C_1, \ldots, C_n) \in \mathcal{D}}(\operatorname{conv} C_1 + \cdots + \operatorname{conv} C_n) = \operatorname{conv} \mathcal{S}$.

For each cell $C = (C_1, \ldots, C_n) \in \mathcal{D}$, we use the notations

$$\operatorname{type} C = (\dim(\operatorname{conv} C_1), \ldots, \dim(\operatorname{conv} C_n)),$$
$$\operatorname{conv} C = \operatorname{conv} C_1 + \cdots + \operatorname{conv} C_n,$$
$$\operatorname{Vol}_n C = \operatorname{Vol}_n(\operatorname{conv} C).$$

Note that the above conditions describe a subdivision of the single polytope $\widetilde{\mathcal{Q}} = \mathcal{Q}_1 + \cdots + \mathcal{Q}_n$ with $\mathcal{Q}_i = \operatorname{conv} \mathcal{S}_i$ for $i = 1, \ldots, n$, with the only special restriction that each cell in the subdivision must be the Minkowski sum of n polytopes whose vertices lie in $\mathcal{S}_1, \ldots, \mathcal{S}_n$ respectively.

While a subdivision of $\widetilde{\mathcal{Q}}$ is important in computing the volume of $\widetilde{\mathcal{Q}}$ (as long as the volume of each sub-polytope is easy to obtain), in studying mixed volume computation, it is more important to find the expression of the volume of the Minkowski sum $\lambda_1 \mathcal{Q}_1 + \cdots + \lambda_n \mathcal{Q}_n$ in relation to the factors $\lambda_1, \ldots, \lambda_n$. For this purpose, a mere subdivision is insufficient. Explicitly, for $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n) \in (\mathbb{R}^+)^n$, we use the notation $\boldsymbol{\lambda} \circ \mathcal{S}$ for the scaled version $(\lambda_1 \mathcal{S}_1, \ldots, \lambda_n \mathcal{S}_n)$. Similarly, for a cell $C = (C_1, \ldots, C_n)$ in a subdivision $\mathcal{D}$ of $\mathcal{S} = (\mathcal{S}_1, \ldots, \mathcal{S}_n)$, we used the notation $\boldsymbol{\lambda} \circ (C_1, \ldots, C_n) = (\lambda_1 C_1, \ldots, \lambda_n C_n)$. An example below will illustrate that a subdivision $\mathcal{D}$ of $\mathcal{S} = (\mathcal{S}_1, \ldots, \mathcal{S}_n)$ does not, in general, extend properly under such "mixed" scaling operation. Namely,

$$\boldsymbol{\lambda} \circ \mathcal{D} := \{\boldsymbol{\lambda} \circ C = (\lambda_1 C_1, \ldots, \lambda_n C_n) \mid C = (C_1, \ldots, C_n) \in \mathcal{D}\}$$

may not be a subdivision of $\boldsymbol{\lambda} \circ \mathcal{S}$. Inevitably, to make a subdivision *scaling invariant*, which turns out to be a critically important property for our mixed volume computation, additional restrictions are necessary:

Definition 2 (Mixed subdivision). A subdivision $\mathcal{D}$ of $\mathcal{S} = (\mathcal{S}_1, \ldots, \mathcal{S}_n)$ is called a **mixed subdivision** if, in addition, it satisfies

(d1): For each cell $C = (C_1, \ldots, C_n) \in \mathcal{D}$, $\sum_{i=1}^{n} \dim(\text{conv } C_i) = n$,

(d2): For any distinct pair of cells $A = (A_1, \ldots, A_n)$ and $B = (B_1, \ldots, B_n)$ in $\mathcal{D}$,

$$(\text{conv } A) \cap (\text{conv } B) = \sum_{i=1}^{n} (\text{conv } A_i \cap \text{conv } B_i).$$

Cells of a mixed subdivision are called **mixed cells**.

As Example 5 below shows, these two conditions are essential in relating the cells in a subdivision and the volume $\text{Vol}_n(\lambda_1 \mathcal{Q}_1 + \cdots + \lambda_n \mathcal{Q}_n)$ by which the mixed volume is defined.

A mixed subdivision of $\mathcal{S} = (\mathcal{S}_1, \ldots, \mathcal{S}_n)$ may be refined via further subdivision of individual components of each cell. It is computationally advantageous (as Equation (10) will show later) to consider the most refined mixed subdivision:

Definition 3 (Fine mixed subdivision). A mixed subdivision $\mathcal{D}$ of $\mathcal{S} = (\mathcal{S}_1, \ldots, \mathcal{S}_n)$ is called a **fine mixed subdivision** if it also satisfies the condition

(e): For each cell $C = (C_1, \ldots, C_n) \in \mathcal{D}$, $\text{conv } C_i$ is a simplex of dimension $\#C_i - 1$ for $i = 1, \ldots, n$.

For mixed subdivisions (and fine mixed subdivisions), we then have the important property of *scaling invariance*:

Proposition 4. *Let $\mathcal{D}$ be a (fine) mixed subdivision of $\mathcal{S} = (\mathcal{S}_1, \ldots, \mathcal{S}_n)$. Then for any $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n) \in (\mathbb{R}^+)^n$, the set*

$$\boldsymbol{\lambda} \circ \mathcal{D} := \{\boldsymbol{\lambda} \circ C = (\lambda_1 C_1, \ldots, \lambda_n C_n) \mid C = (C_1, \ldots, C_n) \in \mathcal{D}\}$$

forms a (fine) mixed subdivision of $\boldsymbol{\lambda} \circ \mathcal{S} = (\lambda_1 \mathcal{S}_1, \ldots, \lambda_n \mathcal{S}_n)$.

This proposition lays the ground for the approach of mixed volume computation we shall discuss in this article. The complete proof of it is somewhat technical, it will be detailed in the next section. In the following example, we shall illustrate the importance of the condition **(d2)**. Without it, the subtlety of nice scaling behavior of the subdivision may disappear.

Example 5 (Subdivision without condition (d2)). With $n = 2$, consider two finite subsets $\mathcal{S}_1 = \{(0,0),(1,0),(0,1)\}$ and $\mathcal{S}_2 = \{(0,0),(1,0),(0,1)\}$ in $\mathbb{R}^2$ and their convex hulls $\mathcal{Q}_1 = \text{conv}\,\mathcal{S}_1$ and $\mathcal{Q}_2 = \text{conv}\,\mathcal{S}_2$ as shown below
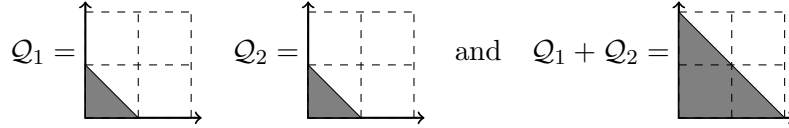
Let $A = (\{(0,0),(1,0),(0,1)\},\{(0,0)\}), B = (\{(0,0),(1,0),(0,1)\},\{(0,1)\})$, and $C = (\{(1,0),(0,1)\},\{(0,0),(1,0)\})$ be three cells of $\mathcal{S} = (\mathcal{S}_1, \mathcal{S}_2)$. Let $\mathcal{D} = \{A, B, C\}$. The convex hull of each cell becomes:
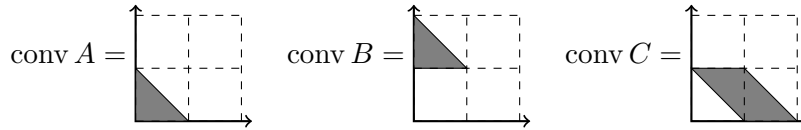
Clearly, $\dim(\text{conv}\,A) = \dim(\text{conv}\,B) = \dim(\text{conv}\,C) = 2$ and the intersection of any two is their common face. Furthermore, the union of the three convex hulls

$$\text{conv}\,A \;\cup\; \text{conv}\,B \;\cup\; \text{conv}\,C \quad = \quad$$
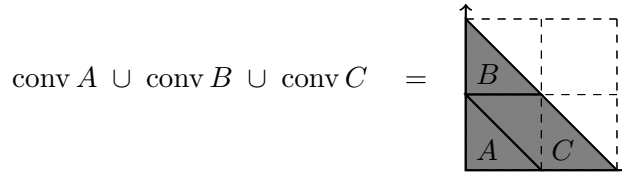
is indeed the entire $\widetilde{\mathcal{Q}} = \mathcal{Q}_1 + \mathcal{Q}_2$. Therefore $\mathcal{D} = \{A, B, C\}$ satisfies the definition of a subdivision (conditions **(a),(b),(c)**). In fact, since the cells A, B, and C are of types $(2,0), (2,0)$, and $(1,1)$ respectively, $\mathcal{D}$ also satisfies condition **(d1)**. But, the failure of condition **(d2)** for subdivision $\mathcal{D}$ can be easily verified. As a consequence, it may not behave favorably under the mixed scaling by $\boldsymbol{\lambda} = (\lambda_1, \lambda_2)$. Namely, for certain scaling $\boldsymbol{\lambda} = (\lambda_1, \lambda_2)$, $\boldsymbol{\lambda} \circ \mathcal{D} = (\boldsymbol{\lambda} \circ A, \boldsymbol{\lambda} \circ B, \boldsymbol{\lambda} \circ C)$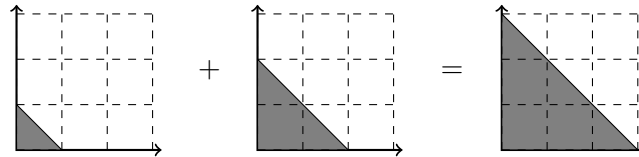 may not be a subdivision of $\boldsymbol{\lambda} \circ \mathcal{S} = (\lambda_1 S_1, \lambda_2 S_2)$. The cells will both separate and overlap as different scaling factors are chosen.

For example, with factors $\boldsymbol{\lambda} = (\lambda_1, \lambda_2) = (1, 2)$, the scaled version of $\widetilde{\mathcal{Q}} = \mathcal{Q}_1 + \mathcal{Q}_2$ is $\boldsymbol{\lambda} \circ \widetilde{\mathcal{Q}} = 1 \cdot \mathcal{Q}_1 + 2 \cdot \mathcal{Q}_2$:

But under the same scaling factors the cells separate with the union of their convex hulls equal to

$$\text{conv}((1,2) \circ A) \ \cup \ \text{conv}((1,2) \circ B) \ \cup \ \text{conv}((1,2) \circ C) \ = \ $$.

It no longer covers the entire $\boldsymbol{\lambda} \circ \widetilde{\mathcal{Q}} = 1 \cdot \mathcal{Q}_1 + 2 \cdot \mathcal{Q}_2$.

Alternatively, with the scaling factor $\boldsymbol{\lambda} = (2,1)$, $\boldsymbol{\lambda} \circ \widetilde{\mathcal{Q}} = 2 \cdot \mathcal{Q}_1 + 1 \cdot \mathcal{Q}_2$ becomes

Under the same scaling, $\text{conv}((2,1) \circ A), \text{conv}((2,1) \circ B), \text{conv}((2,1) \circ C)$ become

respectively. Clearly there are overlaps among the three.

In both cases, the scaled subdivision $\boldsymbol{\lambda} \circ \mathcal{D}$ failed to form a subdivision of $\boldsymbol{\lambda} \circ \widetilde{\mathcal{Q}} = \lambda_1 \mathcal{Q}_1 + \lambda_2 \mathcal{Q}_2$. The problem lies in the fact that the subdivision $\mathcal{D}$ of $\mathcal{S}$ does not satisfy the condition **(d2)**.

Remark 6. The condition **(d2)** was absent when the "mixed subdivision" was originally defined in [16]. It first appeared in [6].

3. Scaling invariance of mixed subdivisions

In this section we will provide a detailed proof of Proposition 4 presented in the last section, that is: *If $\mathcal{D}$ is a (fine) mixed subdivision of $\mathcal{S} = (\mathcal{S}_1, \ldots, \mathcal{S}_n)$, then for $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n) \in (\mathbb{R}^+)^n$,*

$$\boldsymbol{\lambda} \circ \mathcal{D} := \{\boldsymbol{\lambda} \circ C := (\lambda_1 C_1, \ldots, \lambda_n C_n) \ : \ C = (C_1, \ldots, C_n) \in \mathcal{D}\}$$

is a (fine) mixed subdivision of $\boldsymbol{\lambda} \circ \mathcal{S} = (\lambda_1 \mathcal{S}_1, \ldots, \lambda_n \mathcal{S}_n)$.

In the first place, we define the **support function**: For a nonzero vector $\boldsymbol{\alpha} \in \mathbb{R}^n$ and a compact set $P \subset \mathbb{R}^n$ define

$$\mathrm{ht}_{\boldsymbol{\alpha}}\, P := \min_{\boldsymbol{p} \in P} \langle \boldsymbol{p}, \boldsymbol{\alpha} \rangle, \qquad (P)_{\boldsymbol{\alpha}} := \{\boldsymbol{p} \in P : \langle \boldsymbol{p}, \boldsymbol{\alpha} \rangle = \mathrm{ht}_{\boldsymbol{\alpha}}(P)\}.$$

Here the notation "$\mathrm{ht}_{\boldsymbol{\alpha}}$" is used since this function measures the "height" of the set P in the direction given by $\boldsymbol{\alpha}$. It is clear that for $\lambda_i \in \mathbb{R}^+$ and compact sets $P_i \subset \mathbb{R}^n$ for $i = 1, \ldots, n$,

$$\mathrm{ht}_{\boldsymbol{\alpha}}(\lambda_1 P_1 + \cdots + \lambda_n P_n) = \lambda_1 \mathrm{ht}_{\boldsymbol{\alpha}} P_1 + \cdots + \lambda_n \mathrm{ht}_{\boldsymbol{\alpha}} P_n$$
$$(\lambda_1 P_1 + \cdots + \lambda_n P_n)_{\boldsymbol{\alpha}} = \lambda_1 (P_1)_{\boldsymbol{\alpha}} + \cdots + \lambda_n (P_n)_{\boldsymbol{\alpha}}.$$

Furthermore, for compact sets $A, B \subseteq \mathbb{R}^n$,

$$\mathrm{ht}_{\boldsymbol{\alpha}}(A \cup B) = \min\{\mathrm{ht}_{\boldsymbol{\alpha}}(A), \mathrm{ht}_{\boldsymbol{\alpha}}(B)\},$$

and hence

$$(A \cup B)_{\boldsymbol{\alpha}} \subseteq (A)_{\boldsymbol{\alpha}} \cup (B)_{\boldsymbol{\alpha}}.$$

For a finite set $A \subset \mathbb{R}^n$,

$$(\mathrm{conv}\, A)_{\boldsymbol{\alpha}} = \mathrm{conv}\, (A)_{\boldsymbol{\alpha}}.$$

For the rest of this section, let $\mathcal{D}$ be a mixed subdivision of $\mathcal{S} = (\mathcal{S}_1, \ldots, \mathcal{S}_n)$ and $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)$ be positive scaling factors. We now explore the behavior of a mixed subdivision under scaling via the following lemmas.

Lemma 7. *For any cell $C = (C_1, \ldots, C_n) \in \mathcal{D}$, if $\boldsymbol{x} \in \mathrm{conv}(\boldsymbol{\lambda} \circ C)$ then there exists a unique n-tuple $(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n)$ with $\boldsymbol{x}_i \in C_i$ for all i such that $\boldsymbol{x} = \lambda_1 \boldsymbol{x}_1 + \cdots + \lambda_n \boldsymbol{x}_n$.*

Proof. Suppose there are pairs $\boldsymbol{x}_1, \boldsymbol{x}_1' \in \mathrm{conv}\, C_1, \ldots, \boldsymbol{x}_n, \boldsymbol{x}_n' \in \mathrm{conv}\, C_n$ such that

$$\boldsymbol{x} = \lambda_1 \boldsymbol{x}_1 + \cdots + \lambda_n \boldsymbol{x}_n = \lambda_1 \boldsymbol{x}_1' + \cdots + \lambda_n \boldsymbol{x}_n'.$$

Then $\lambda_1 (\boldsymbol{x}_1' - \boldsymbol{x}_1) + \cdots + \lambda_n (\boldsymbol{x}_n' - \boldsymbol{x}_n) = \boldsymbol{0}$. For each $i = 1, \ldots, n$, let $L_i = \mathrm{span}\{C_i - \boldsymbol{x}_i\}$, then $\dim (L_i) = \dim (\mathrm{conv}\, C_i)$. Clearly $\mathrm{conv}\, C \subset \boldsymbol{x} + (L_1 + \cdots + L_n)$, so $\dim (\mathrm{conv}\, C) \leq \dim(L_1 + \cdots + L_n)$. By condition **(d1)** in the

definition of mixed subdivision,

$$\dim\left(\operatorname{conv} C\right) = n = \sum_{i=1}^{n} \dim\left(\operatorname{conv} C_i\right) = \sum_{i=1}^{n} \dim\left(L_i\right).$$

Therefore the subspaces $L_1, \ldots, L_n$ must be linearly independent. But $\boldsymbol{x}_i' - \boldsymbol{x}_i \in L_i$ for each $i = 1, \ldots, n$ and $\lambda_1(\boldsymbol{x}_1' - \boldsymbol{x}_1) + \cdots + \lambda_n(\boldsymbol{x}_n' - \boldsymbol{x}_n) = \boldsymbol{0}$, so each $\lambda_i(\boldsymbol{x}_i' - \boldsymbol{x}_i)$ must be zero and hence $\boldsymbol{x}_i' = \boldsymbol{x}_i$ for each $i = 1, \ldots, n$.   □

Taking $\boldsymbol{\lambda} = (1, \ldots, 1)$, this lemma implies that for any cell $C = (C_1, \ldots, C_n) \in \mathcal{D}$, each point $\boldsymbol{x} \in \operatorname{conv} C$ has a unique representation as a sum $\boldsymbol{x} = \boldsymbol{x}_1 + \cdots + \boldsymbol{x}_n$ with $\boldsymbol{x}_i \in \operatorname{conv} C_i$ for $i = 1, \ldots, n$. So, the scaling map $\phi_{\boldsymbol{\lambda}}^{\mathcal{D}} : \operatorname{conv} C \to \operatorname{conv}(\boldsymbol{\lambda} \circ C)$ given by

$$(1) \qquad \phi_{\boldsymbol{\lambda}}^{\mathcal{D}}(\boldsymbol{x}) := \lambda_1 \boldsymbol{x}_1 + \cdots + \lambda_n \boldsymbol{x}_n, \qquad \text{for} \ \ \boldsymbol{x} = \boldsymbol{x}_1 + \cdots + \boldsymbol{x}_n$$

is well defined as a nonsingular affine map.

Lemma 8. *Let* $A = (A_1, \ldots, A_n), B = (B_1, \ldots, B_n) \in \mathcal{D}$. *If* $\boldsymbol{a}_1 + \cdots + \boldsymbol{a}_n \in \operatorname{conv} A \cap \operatorname{conv} B$ *with* $\boldsymbol{a}_1 \in \operatorname{conv} A_1, \ldots, \boldsymbol{a}_n \in \operatorname{conv} A_n$ *then* $\boldsymbol{a}_1 \in \operatorname{conv} B_1, \ldots, \boldsymbol{a}_n \in \operatorname{conv} B_n$.

Proof. Since $\boldsymbol{a}_1 + \cdots + \boldsymbol{a}_n \in \operatorname{conv} A \cap \operatorname{conv} B = \sum_{i=1}^{n}(\operatorname{conv} A_i \cap \operatorname{conv} B_i)$ by condition **(d2)** in the definition of mixed subdivision, and $\operatorname{conv} A_i \cap \operatorname{conv} B_i \subseteq \operatorname{conv} A_i$ for each $i = 1, \ldots, n$, hence $\boldsymbol{a}_i \in \operatorname{conv} A_i \cap \operatorname{conv} B_i$ because the representation of $\boldsymbol{a}_1 + \cdots + \boldsymbol{a}_n$ in $\operatorname{conv} A$ must be unique by Lemma 7. Therefore, for each $i = 1, \ldots, n$, $\boldsymbol{a}_i \in \operatorname{conv} B_i$.   □

By this lemma, for any point on the common face of two cells, its representations in both cells must be identical, which, along with Lemma 7, implies that every point $\boldsymbol{x} \in \widetilde{\mathcal{Q}} = \mathcal{Q}_1 + \cdots + \mathcal{Q}_n$ has a unique representation of the form $\boldsymbol{x}_1 + \cdots + \boldsymbol{x}_n$ with $\boldsymbol{x}_i \in \mathcal{Q}_i$. Therefore the scaling map (1) can be extended to a well defined continuous map $\phi_{\boldsymbol{\lambda}}^{\mathcal{D}} : \widetilde{\mathcal{Q}} \to \boldsymbol{\lambda} \circ \widetilde{\mathcal{Q}} = \lambda_1 \mathcal{Q}_1 + \cdots + \lambda_n \mathcal{Q}_n$ by

$$(2) \qquad\qquad \phi_{\boldsymbol{\lambda}}^{\mathcal{D}}(\boldsymbol{x}) = \lambda_1 \boldsymbol{x}_1 + \cdots + \lambda_n \boldsymbol{x}_n$$

where $\boldsymbol{x} = \boldsymbol{x}_1 + \cdots + \boldsymbol{x}_n$ with $\boldsymbol{x}_i \in \mathcal{Q}_i$ is the unique representation of $\boldsymbol{x}$ given by the mixed subdivision $\mathcal{D}$. A direct consequence is that under scaling neighboring cells remain neighbors:

Lemma 9. *For* $A, B \in \mathcal{D}$, *if* $(\operatorname{conv} A) \cap (\operatorname{conv} B) \neq \varnothing$, *then* $\operatorname{conv}(\boldsymbol{\lambda} \circ A) \cap \operatorname{conv}(\boldsymbol{\lambda} \circ B) \neq \varnothing$.

Proof. For $A = (A_1, \ldots, A_n)$ and $B = (B_1, \ldots, B_n)$, let $\boldsymbol{a} \in \operatorname{conv} A \cap \operatorname{conv} B$, then $\boldsymbol{a} = \boldsymbol{a}_1 + \cdots + \boldsymbol{a}_n$ for some $\boldsymbol{a}_i \in \operatorname{conv} A_i$ for $i = 1, \ldots, n$. But by the previous lemma, each $\boldsymbol{a}_i \in \operatorname{conv} B_i$. Thus for $i = 1, \ldots, n$, $\lambda_i \boldsymbol{a}_i$ is in both $\operatorname{conv} \lambda_i A_i$ and $\operatorname{conv} \lambda_i B_i$. Therefore $\phi_{\boldsymbol{\lambda}}^{\mathcal{D}}(\boldsymbol{a}) \in \operatorname{conv}(\boldsymbol{\lambda} \circ A) \cap \operatorname{conv}(\boldsymbol{\lambda} \circ B)$, and it cannot be empty. $\qquad\square$

Defined in terms of the mixed subdivision $\mathcal{D}$, the scaling map $\phi_{\boldsymbol{\lambda}}^{\mathcal{D}}$ is well defined and continuous on all of $\widetilde{\mathcal{Q}} = \mathcal{Q}_1 + \cdots + \mathcal{Q}_n$.

Lemma 10. *Let* $A = (A_1, \ldots, A_n), B = (B_1, \ldots, B_n) \in \mathcal{D}$ *with* $\operatorname{conv} A \cap \operatorname{conv} B \neq \varnothing$. *If* $\boldsymbol{a} \in \operatorname{conv} A$, $\boldsymbol{b} \in \operatorname{conv} B$ *and* $\boldsymbol{a} \neq \boldsymbol{b}$, *then* $\phi_{\boldsymbol{\lambda}}^{\mathcal{D}}(\boldsymbol{a}) \neq \phi_{\boldsymbol{\lambda}}^{\mathcal{D}}(\boldsymbol{b})$.

Proof. If both $\boldsymbol{a}$ and $\boldsymbol{b}$ are in $\operatorname{conv} A \cap \operatorname{conv} B$, then the agreement of $\phi_{\boldsymbol{\lambda}}^{\mathcal{D}}(\boldsymbol{a})$ and $\phi_{\boldsymbol{\lambda}}^{\mathcal{D}}(\boldsymbol{b})$, both in $\operatorname{conv}(\boldsymbol{\lambda} \circ A)$, would contradict to the unique representation of $\phi_{\boldsymbol{\lambda}}^{\mathcal{D}}(\boldsymbol{a})$ provided by Lemma 7.

On the other hand, if $\boldsymbol{a} \notin \operatorname{conv} A \cap \operatorname{conv} B$, let $\boldsymbol{r} = \boldsymbol{r}_1 + \cdots + \boldsymbol{r}_n$ be any point in $(\operatorname{conv} A) \cap (\operatorname{conv} B)$. Then, since $(\operatorname{conv} A) \cap (\operatorname{conv} B)$ is a face of both $\operatorname{conv} A$ and $\operatorname{conv} B$, passing through $\boldsymbol{r}$ there exists a hyperplane H defined by $\langle \boldsymbol{\alpha}, \bullet \rangle = h$ for some $\boldsymbol{\alpha} \in \mathbb{R}^n$ and $h \in \mathbb{R}$ such that $\operatorname{int}(\operatorname{conv} A)$ and $\operatorname{int}(\operatorname{conv} B)$ are in $H^- := \{\boldsymbol{x} \in \mathbb{R}^n \mid \langle \boldsymbol{\alpha}, \boldsymbol{x} \rangle < h\}$ and $H^+ := \{\boldsymbol{x} \in \mathbb{R}^n \mid \langle \boldsymbol{\alpha}, \boldsymbol{x} \rangle > h\}$ respectively. Also, $\boldsymbol{r} \in (\operatorname{conv} A)_{\boldsymbol{\alpha}}$ and $\boldsymbol{r} \in (\operatorname{conv} B)_{-\boldsymbol{\alpha}}$, hence by linearity, $\boldsymbol{r}_i \in (\operatorname{conv} A_i)_{\boldsymbol{\alpha}}$ and $\boldsymbol{r}_i \in (\operatorname{conv} B_i)_{-\boldsymbol{\alpha}}$ for each $i = 1, \ldots, n$. That is, with $\boldsymbol{a} = \boldsymbol{a}_1 + \cdots + \boldsymbol{a}_n$ and $\boldsymbol{b} = \boldsymbol{b}_1 + \cdots + \boldsymbol{b}_n$,

$$\langle \boldsymbol{\alpha}, \boldsymbol{a}_i \rangle > \langle \boldsymbol{\alpha}, \boldsymbol{r}_i \rangle \geq \langle \boldsymbol{\alpha}, \boldsymbol{b}_i \rangle$$

for each $i = 1, \ldots, n$. Therefore

$$\langle \boldsymbol{\alpha}, \boldsymbol{\lambda} \circ \boldsymbol{a} \rangle = \sum_{i=1}^{n} \lambda_i \langle \boldsymbol{\alpha}, \boldsymbol{a}_i \rangle > \sum_{i=1}^{n} \lambda_i \langle \boldsymbol{\alpha}, \boldsymbol{r}_i \rangle \geq \sum_{i=1}^{n} \lambda_i \langle \boldsymbol{\alpha}, \boldsymbol{b}_i \rangle = \langle \boldsymbol{\alpha}, \boldsymbol{\lambda} \circ \boldsymbol{b} \rangle.$$

So, $\boldsymbol{\lambda} \circ \boldsymbol{a} \neq \boldsymbol{\lambda} \circ \boldsymbol{b}$, that is, $\phi_{\boldsymbol{\lambda}}^{\mathcal{D}}(\boldsymbol{a}) \neq \phi_{\boldsymbol{\lambda}}^{\mathcal{D}}(\boldsymbol{b})$. $\qquad\square$

Consequently, fix a cell $A \in \mathcal{D}$, for any neighboring cell $B \in \mathcal{D}$, the continuous map $\phi_{\boldsymbol{\lambda}}^{\mathcal{D}}$ is also one-to-one on $(\operatorname{conv} A) \cup (\operatorname{conv} B)$.

With these lemmas ready, we now prove Proposition 4.

Proof of Proposition 4. Let $\mathcal{D}$ be a (fine) mixed subdivision of the n-tuple $\mathcal{S} = (\mathcal{S}_1, \ldots, \mathcal{S}_n)$, and $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n) \in (\mathbb{R}^+)^n$. For the proof of $\boldsymbol{\lambda} \circ \mathcal{D} := \{\boldsymbol{\lambda} \circ C = (\lambda_1 C_1, \ldots, \lambda_n C_n) : C = (C_1, \ldots, C_n) \in \mathcal{D}\}$ being a (fine) mixed subdivision of $\boldsymbol{\lambda} \circ \mathcal{S} = (\lambda_1 \mathcal{S}_1, \ldots, \lambda_n \mathcal{S}_n)$, conditions **(a), (d1),** and **(e)** in

Definitions 1, 2, and 3 are clearly valid since for a cell $C = (C_1, \ldots, C_n) \in \mathcal{D}$, the scaling $\operatorname{conv} C_i \mapsto \operatorname{conv}(\lambda_i C_i)$, for each $i = 1, \ldots, n$ is a nonsingular affine transformation which preserves dimensions of affine spaces. Below we verify that conditions **(b)**,**(c)**, and **(d2)** hold.

(b) For condition **(b)**, let $A = (A_1, \ldots, A_n)$ and $B = (B_1, \ldots, B_n)$ be two cells in $\mathcal{D}$. We first show that $\operatorname{conv}(\boldsymbol{\lambda} \circ A) \cap \operatorname{conv}(\boldsymbol{\lambda} \circ B) \neq \varnothing$ yields $(\operatorname{conv} A) \cap (\operatorname{conv} B) \neq \varnothing$. Suppose that $\operatorname{conv} A \cap \operatorname{conv} B = \varnothing$. For $\boldsymbol{x} \in \operatorname{conv} \widetilde{\mathcal{Q}}$ and $s \in [0, 1]$, define

$$h(\boldsymbol{x}, s) := \phi^{\mathcal{D}}_{[(1-s)\boldsymbol{\lambda}+s\mathbf{1}]}(\boldsymbol{x})$$

where $\mathbf{1} = (1, \ldots, 1)$. By the continuity we established for the map $\phi^{\mathcal{D}}_{\boldsymbol{\lambda}} : \widetilde{\mathcal{Q}} \to \boldsymbol{\lambda} \circ \widetilde{\mathcal{Q}}$ before, $h(\boldsymbol{x}, s)$ is continuous for each $s \in [0, 1]$. Furthermore, since $H(\boldsymbol{x}, 1) = \boldsymbol{x}$ and $H(\boldsymbol{x}, 0) = \phi^{\mathcal{D}}_{\boldsymbol{\lambda}}(\boldsymbol{x})$, so

$$h(\operatorname{conv} A, 1) \cap H(\operatorname{conv} B, 1) = (\operatorname{conv} A) \cap (\operatorname{conv} B) = \varnothing \quad \text{and}$$
$$h(\operatorname{conv} A, 0) \cap H(\operatorname{conv} B, 0) = \operatorname{conv}(\boldsymbol{\lambda} \circ A) \cap \operatorname{conv}(\boldsymbol{\lambda} \circ B) \neq \varnothing.$$

Thus, there exits $s_0 \in (0, 1)$, for which

$$h(\operatorname{conv} A, s_0) \cap H(\operatorname{conv} B, s_0) \neq \varnothing \quad \text{and}$$
$$h(\operatorname{conv} A, s) \cap H(\operatorname{conv} B, s) = \varnothing \quad \text{for } 0 < s_0 < s \leq 1.$$

Let $N := \{C \in \mathcal{D} : \operatorname{conv} C \cap \operatorname{conv} A \neq \varnothing\}$, and

$$G = \left(\bigcup_{C \in N} \operatorname{conv} C \right), \qquad T = \partial G \setminus \partial(\operatorname{conv} \mathcal{S}).$$

Geometrically, T may be considered a "shell" that separates $\operatorname{conv} A$ and $\operatorname{conv} B$. By Lemma 9, neighboring cells remain neighbors after the mixed scaling, so $h(T, s)$ remains connected for all $s \in [0, 1]$. Therefore as s decreases from 1, since $h(\operatorname{conv} A, s)$ and $h(\operatorname{conv} B, s)$ will meet at s_0, $h(\operatorname{conv} A, s)$ and $h(T, s)$ must intersect at some $s_1 \in [s_0, 1]$. But T consists of boundaries of neighbors of A, so there is a neighbor cell C of A in $\mathcal{D}$ and a point $\boldsymbol{x} \in \operatorname{conv} C \setminus \operatorname{conv} A$ such that $\phi^{\mathcal{D}}_{[(1-s_1)\boldsymbol{\lambda}+s\mathbf{1}]}(\boldsymbol{x}) \in \phi^{\mathcal{D}}_{[(1-s_1)\boldsymbol{\lambda}+s\mathbf{1}]}(\operatorname{conv} A)$. This violates Lemma 10. Therefore $(\operatorname{conv} A) \cap (\operatorname{conv} B) \neq \varnothing$.

Now, since $(\mathrm{conv}\,A) \cap (\mathrm{conv}\,B) \neq \varnothing$ is a common face of both conv A and conv B, by the hyperplane separation theorem, there exists a nonzero vector $\boldsymbol{\alpha}$ such that

$$(\mathrm{conv}\,A) \cap (\mathrm{conv}\,B) = (\mathrm{conv}\,A)_{\boldsymbol{\alpha}} = (\mathrm{conv}\,B)_{-\boldsymbol{\alpha}}.$$

We claim that a hyperplane defined by $\boldsymbol{\alpha}$ separates $\mathrm{conv}(\boldsymbol{\lambda} \circ A)$ and $\mathrm{conv}(\boldsymbol{\lambda} \circ B)$. To prove this, let $\boldsymbol{a}' \in (\mathrm{conv}\,\boldsymbol{\lambda} \circ A)_{\boldsymbol{\alpha}}$, then $\boldsymbol{a}' = \lambda_1 \boldsymbol{a}_1 + \cdots + \lambda_n \boldsymbol{a}_n$ for some $\boldsymbol{a}_1 \in \mathrm{conv}\,A_1, \ldots, \boldsymbol{a}_n \in \mathrm{conv}\,A_n$. By linearity, $\boldsymbol{a}_i \in (\mathrm{conv}\,A_i)_{\boldsymbol{\alpha}}$ for each $i = 1, \ldots, n$, and hence $\boldsymbol{a}_1 + \cdots + \boldsymbol{a}_n \in (\mathrm{conv}\,A)_{\boldsymbol{\alpha}} = \mathrm{conv}\,A \cap \mathrm{conv}\,B$. By Lemma 8, $\boldsymbol{a}_i \in \mathrm{conv}\,B_i$ for each i, and therefore $\boldsymbol{a}' = \lambda_1 \boldsymbol{a}_1 + \cdots + \lambda_n \boldsymbol{a}_n \in \mathrm{conv}(\boldsymbol{\lambda} \circ B)$. This shows $(\mathrm{conv}\,\boldsymbol{\lambda} \circ A)_{\boldsymbol{\alpha}} \subset \mathrm{conv}\,\boldsymbol{\lambda} \circ B$. Secondly, if $\boldsymbol{a}' \in \mathrm{conv}(\boldsymbol{\lambda} \circ A) \setminus (\mathrm{conv}\,\boldsymbol{\lambda} \circ A)_{\boldsymbol{\alpha}}$, by Lemma 7 there exists a unique list $\boldsymbol{a}_1 \in \mathrm{conv}\,A_1, \ldots, \boldsymbol{a}_n \in \mathrm{conv}\,A_n$ such that $\boldsymbol{a}' = \lambda_1 \boldsymbol{a}_1 + \cdots + \lambda_n \boldsymbol{a}_n$. Then $\boldsymbol{a}_1 + \cdots + \boldsymbol{a}_n \in \mathrm{conv}\,A \setminus \mathrm{conv}\,B$. By Lemma 10, $\boldsymbol{a}' \notin \mathrm{conv}(\boldsymbol{\lambda} \circ B)$. Combining those two parts shows that the subset of $\mathrm{conv}(\boldsymbol{\lambda} \circ A)$ inside $\mathrm{conv}(\boldsymbol{\lambda} \circ B)$ is exactly $(\mathrm{conv}(\boldsymbol{\lambda} \circ A))_{\boldsymbol{\alpha}}$.

By a symmetric argument, the only subset of $\mathrm{conv}(\boldsymbol{\lambda} \circ B)$ inside $\mathrm{conv}(\boldsymbol{\lambda} \circ A)$ is exactly $(\mathrm{conv}(\boldsymbol{\lambda} \circ B))_{-\boldsymbol{\alpha}}$. Therefore $\mathrm{conv}(\boldsymbol{\lambda} \circ A) \cap \mathrm{conv}(\boldsymbol{\lambda} \circ B) = (\mathrm{conv}(\boldsymbol{\lambda} \circ A))_{\boldsymbol{\alpha}} = (\mathrm{conv}(\boldsymbol{\lambda} \circ B))_{-\boldsymbol{\alpha}}$ is a common face of both.

(c) We need to show $M := \bigcup_{C \in \mathcal{D}} \mathrm{conv}(\boldsymbol{\lambda} \circ C) = \boldsymbol{\lambda} \circ \widetilde{\mathcal{Q}} := \lambda_1 \mathcal{Q}_1 + \cdots + \lambda_n \mathcal{Q}_n$ where $\mathcal{Q}_i = \mathrm{conv}\,\mathcal{S}_i$ for $i = 1, \ldots, n$. One direction of the containment is obvious: since for each $C \in \mathcal{D}$, $\mathrm{conv}(\boldsymbol{\lambda} \circ C) = \boldsymbol{\lambda} \circ \mathrm{conv}\,C \subseteq \boldsymbol{\lambda} \circ \widetilde{\mathcal{Q}}$, the union is still in $\boldsymbol{\lambda} \circ \widetilde{\mathcal{Q}}$. For the other direction, suppose $M \subsetneq \boldsymbol{\lambda} \circ \widetilde{\mathcal{Q}}$, then M has boundary points outside $\partial(\boldsymbol{\lambda} \circ \widetilde{\mathcal{Q}})$. Let $\boldsymbol{q} \in \partial M \setminus \partial(\boldsymbol{\lambda} \circ \widetilde{\mathcal{Q}})$. Then, $\boldsymbol{q}$ is in

$$\partial \bigcup_{C \in \mathcal{D}} \mathrm{conv}(\boldsymbol{\lambda} \circ C) \setminus \partial(\boldsymbol{\lambda} \circ \widetilde{\mathcal{Q}}) \subseteq \bigcup_{C \in \mathcal{D}} \partial(\mathrm{conv}(\boldsymbol{\lambda} \circ C)) \setminus \partial(\boldsymbol{\lambda} \circ \widetilde{\mathcal{Q}}),$$

and there exists a cell $C = (C_1, \ldots, C_n) \in \mathcal{D}$ such that $\boldsymbol{q} \in \partial(\mathrm{conv}(\boldsymbol{\lambda} \circ C))$. Without loss of generality, assume $\boldsymbol{q}$ lies on a facet of $\mathrm{conv}(\boldsymbol{\lambda} \circ C)$ but not on any other of its faces. By Lemma 7, there is a unique list $\boldsymbol{q}_1 \in \mathrm{conv}\,C_1, \ldots, \boldsymbol{q}_n \in \mathrm{conv}\,C_n$ such that $\boldsymbol{q} = \lambda_1 \boldsymbol{q}_1 + \cdots + \lambda_n \boldsymbol{q}_n$. Then $\boldsymbol{q}_1 + \cdots + \boldsymbol{q}_n$ must be on a facet of conv C but not on any other of its faces.

If $\boldsymbol{q}_1 + \cdots + \boldsymbol{q}_n \in \partial\widetilde{\mathcal{Q}}$ then $\boldsymbol{q}_1 + \cdots + \boldsymbol{q}_n \in \mathcal{Q}_{\boldsymbol{\alpha}} = (\mathcal{Q}_1)_{\boldsymbol{\alpha}} + \cdots + (\mathcal{Q}_n)_{\boldsymbol{\alpha}}$ for some nonzero $\boldsymbol{\alpha} \in \mathbb{R}^n$ and hence

$$\boldsymbol{q} = \lambda_1 \boldsymbol{q}_1 + \cdots + \lambda_n \boldsymbol{q}_n$$
$$\in \lambda_1 (\mathcal{Q}_1)_{\boldsymbol{\alpha}} + \cdots + \lambda_n (\mathcal{Q}_n)_{\boldsymbol{\alpha}}$$
$$= (\boldsymbol{\lambda} \circ \widetilde{\mathcal{Q}})_{\boldsymbol{\alpha}}$$
$$\subset \partial(\boldsymbol{\lambda} \circ \widetilde{\mathcal{Q}}),$$

a contradiction. So, $\boldsymbol{q}_1 + \cdots + \boldsymbol{q}_n \notin \partial \widetilde{\mathcal{Q}}$. Therefore $\boldsymbol{q}_1 + \cdots + \boldsymbol{q}_n$ must be on a common facet of $\operatorname{conv} C$ and $\operatorname{conv} C'$ for some other cell $C' = (C'_1, \ldots, C'_n)$. By Lemma 8, $\boldsymbol{q}_1 \in \operatorname{conv} C'_1, \ldots, \boldsymbol{q}_n \in \operatorname{conv} C'_n$, and therefore $\boldsymbol{q} = \lambda_1 \boldsymbol{q}_1 + \cdots + \lambda_n \boldsymbol{q}_n$ is in both $\operatorname{conv}(\boldsymbol{\lambda} \circ C)$ and $\operatorname{conv}(\boldsymbol{\lambda} \circ C')$. Indeed, $\boldsymbol{q}$ is in the interior of $\operatorname{conv}(\boldsymbol{\lambda} \circ C) \cup \operatorname{conv}(\boldsymbol{\lambda} \circ C')$, contradicting to the assumption that $\boldsymbol{q} \in \partial M$. Therefore we can only conclude that $M = \boldsymbol{\lambda} \circ \widetilde{\mathcal{Q}}$.

(d2) Let $A = (A_1, \ldots, A_n)$ and $B = (B_1, \ldots, B_n)$ be cells in $\mathcal{D}$. By condition **(b)**, $\operatorname{conv}(\boldsymbol{\lambda} \circ A) \cap \operatorname{conv}(\boldsymbol{\lambda} \circ B)$, if nonempty, is a common face of both of them, i.e., the intersection is of the form $(\operatorname{conv}(\boldsymbol{\lambda} \circ A))_{\boldsymbol{\alpha}}$ for $\boldsymbol{\alpha} \in \mathbb{R}^n$. So

$$\operatorname{conv}(\boldsymbol{\lambda} \circ A) \cap \operatorname{conv}(\boldsymbol{\lambda} \circ B) = (\operatorname{conv}(\boldsymbol{\lambda} \circ A))_{\boldsymbol{\alpha}}$$
$$= \left(\sum_{i=1}^{n} \operatorname{conv} \lambda_i A_i \right)_{\boldsymbol{\alpha}}$$
$$= \sum_{i=1}^{n} \lambda_i \operatorname{conv}(A_i)_{\boldsymbol{\alpha}}$$
$$= \sum_{i=1}^{n} \lambda_i (\operatorname{conv} A_i \cap \operatorname{conv} B_i).$$

$\square$

4. Computing mixed volume via fine mixed subdivisions

Once a fine mixed subdivision $\mathcal{D}$ of $\mathcal{S} = (\mathcal{S}_1, \ldots, \mathcal{S}_n)$ is available, the mixed volume $\mathcal{M}(\mathcal{Q}_1, \ldots, \mathcal{Q}_n)$ with $\mathcal{Q}_i = \operatorname{conv} \mathcal{S}_i$ can now be computed. This section describes the concrete procedure and provides its theoretical justification.

We begin with the volume change of a cell $C \in \mathcal{D}$ under mixed scaling by $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)$.

Proposition 11 (Cell volume under mixed scaling). *If $\mathcal{D}$ is a fine mixed subdivision, and a cell $C = (C_1, \ldots, C_n) \in \mathcal{D}$ is of type $(k_1, \ldots, k_n)$, then for $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n) \in (\mathbb{R}^+)^n$*

$$(3) \qquad \mathrm{Vol}_n(\boldsymbol{\lambda} \circ C) = \lambda_1^{k_1} \ldots \lambda_n^{k_n} \, \mathrm{Vol}_n(C).$$

Proof. Recall that for a mixed cell $C = (C_1, \ldots, C_n)$ in the fine mixed subdivision $\mathcal{D}$ with $\mathrm{type}(C) = (k_1, \ldots, k_n)$, where $k_i = \dim(\mathrm{conv}\, C_i)$, we have, by definition, $k_1 + \cdots + k_n = n$, and each $\mathrm{conv}\, C_i$ is a simplex of dimension $k_i = \#C_i - 1$. Write $C_i = \{\boldsymbol{c}_0^i, \ldots, \boldsymbol{c}_{k_i}^i\}$ for each $i = 1, \ldots, n$. We shall construct an explicit parametrization of points in $\mathrm{conv}(\boldsymbol{\lambda} \circ C)$.

Any point $\boldsymbol{x} \in \mathrm{conv}(\boldsymbol{\lambda} \circ C)$ can be written, by Lemma 7, as $\lambda_1 \boldsymbol{x}_1 + \cdots + \lambda_n \boldsymbol{x}_n$ for a unique list of $\boldsymbol{x}_i \in \mathrm{conv}\, C_i$ for $i = 1, \ldots, n$. For a fixed i, the point $\boldsymbol{x}_i$ can, in turn, be written as a convex combination of $\{\boldsymbol{c}_0^i, \ldots, \boldsymbol{c}_{k_i}^i\}$, i.e., there exist $t_{i,0}, \ldots, t_{i,k_i} \geq 0$ and $\sum_{j=0}^{k_i} t_{i,j} = 1$ such that $\boldsymbol{x}_i = \sum_{j=0}^{k_i} t_{i,j} \boldsymbol{c}_j^i$. Therefore

$$\boldsymbol{x} = \sum_{i=1}^{n} \left(\lambda_i \sum_{j=0}^{k_i} t_{i,j} \boldsymbol{c}_j^i \right).$$

Rearranging terms yields

$$(4) \qquad \boldsymbol{x} = \sum_{i=1}^{n} \lambda_i \left[\sum_{j=0}^{k_i} t_{i,j} \boldsymbol{c}_0^i + \sum_{j=1}^{k_i} t_{i,j} (\boldsymbol{c}_j^i - \boldsymbol{c}_0^i) \right] = \sum_{i=1}^{n} \lambda_i \left[\boldsymbol{c}_0^i + \sum_{j=1}^{k_i} t_{i,j} (\boldsymbol{c}_j^i - \boldsymbol{c}_0^i) \right].$$

For each $i = 1, \ldots, n$, define the $n \times k_i$ (empty when $k_i = 0$) matrix

$$(5) \qquad V(C_i) := \begin{bmatrix} \boldsymbol{c}_1^i - \boldsymbol{c}_0^i & \cdots & \boldsymbol{c}_{k_i}^i - \boldsymbol{c}_0^i \end{bmatrix}.$$

With these, define the block matrices

$$(6) \qquad V(C) = \begin{bmatrix} V(C_1) & \cdots & V(C_n) \end{bmatrix}$$
$$(7) \qquad V_{\boldsymbol{\lambda}}(C) = \begin{bmatrix} \lambda_1 \cdot V(C_1) & \cdots & \lambda_n \cdot V(C_n) \end{bmatrix}$$

which are both of size $n \times (k_1 + \cdots + k_n) = n \times n$ (if $k_i = 0$ then the block $V(C_i)$ does not appear). We may now rewrite (4) as

$$(8) \qquad \boldsymbol{x} = \left(\sum_{i=1}^{n} \boldsymbol{c}_0^i \right) + V(C) \cdot \begin{bmatrix} \boldsymbol{t}_1 \\ \vdots \\ \boldsymbol{t}_n \end{bmatrix} \quad \text{where } \boldsymbol{t}_i = \begin{bmatrix} t_{i,1} \\ \vdots \\ t_{i,k_i} \end{bmatrix} \text{ for } i = 1, \ldots, n.$$

Again, when $k_i = 0$, $\boldsymbol{t}_i$ will be absent in $(\boldsymbol{t}_1, \ldots, \boldsymbol{t}_n)^T$. Recall that for each i, $t_{i,0} \geq 0, \ldots, t_{i,k_i} \geq 0$ and $\sum_{j=0}^{k_i} t_{i,j} = 1$, hence $0 \leq \sum_{j=1}^{k_i} t_{i,j} \leq 1$. So, each

$\boldsymbol{t}_i = (t_{i,1}, \ldots, t_{i,k_i})^T$ is a point in the standard k_i-simplex Δ_{k_i} (which would be a single point if $k_i = 0$). Thus, the affine map $T : \Delta_{k_1} \times \cdots \times \Delta_{k_n} \to \mathbb{R}^n$ given by

$$
(9) \qquad T(\boldsymbol{t}_1, \ldots, \boldsymbol{t}_n) = \left(\sum_{i=1}^{n} \lambda_i \boldsymbol{c}_0^i \right) + V(C) \cdot \begin{bmatrix} \boldsymbol{t}_1 \\ \vdots \\ \boldsymbol{t}_n \end{bmatrix}
$$

makes every $\boldsymbol{x} \in \operatorname{conv}(\boldsymbol{\lambda} \circ C)$ a point in the image of T. It is clear that the converse is also true: For every point $(\boldsymbol{t}_1, \ldots, \boldsymbol{t}_n) \in \Delta_{k_1} \times \cdots \times \Delta_{k_n}$, $T(\boldsymbol{t}_1, \ldots, \boldsymbol{t}_n)$ lies in $\operatorname{conv}(\boldsymbol{\lambda} \circ C)$. Furthermore, by the uniqueness of representation given by Lemma 7, T is also one-to-one. Therefore, $\operatorname{conv}(\boldsymbol{\lambda} \circ C)$ is precisely the image of $\Delta_{k_1} \times \cdots \times \Delta_{k_n}$ under the affine transformation T, and hence

$$
\operatorname{Vol}_n(\boldsymbol{\lambda} \circ C) = \operatorname{Vol}_n(\operatorname{conv}(\boldsymbol{\lambda} \circ C)) = |\det V_{\boldsymbol{\lambda}}(C)| \cdot \operatorname{Vol}_n(\Delta_{k_1} \times \cdots \times \Delta_{k_n})
$$

where $V_{\boldsymbol{\lambda}}(C)$, defined in (7), is the Jacobian matrix of T. Since $\operatorname{Vol}_n(\Delta_{k_i}) = \dfrac{1}{k_i!}$, so $\operatorname{Vol}_n(\Delta_{k_1} \times \cdots \times \Delta_{k_n}) = \dfrac{1}{k_1! \cdots k_n!}$. By (7),

$$
\begin{aligned}
\det V_{\boldsymbol{\lambda}}(C) &= \det \begin{bmatrix} \lambda_1 \cdot V_1(C) & \cdots & \lambda_n \cdot V_n(C) \end{bmatrix} \\
&= \lambda_1^{k_1} \cdots \lambda_n^{k_n} \det \begin{bmatrix} V_1(C) & \cdots & V_n(C) \end{bmatrix} \\
&= \lambda_1^{k_1} \cdots \lambda_n^{k_n} \det V(C)
\end{aligned}
$$

since each $V_i(C)$ is a block of size $n \times k_i$. It follows that

$$
\operatorname{Vol}_n(\boldsymbol{\lambda} \circ C) = \frac{\lambda_1^{k_1} \cdots \lambda_n^{k_n}}{k_1! \cdots k_n!} |\det V(C)|.
$$

Taking $\lambda = (1, \ldots, 1)$ yields the special case

$$
(10) \qquad \operatorname{Vol}_n(C) = \frac{1}{k_1! \cdots k_n!} \cdot |\det V(C)|.
$$

The assertion of the proposition thus follows by comparing the two equations above. $\qquad\square$

Proposition 12. *If $\mathcal{D}$ is a fine mixed subdivision of $\mathcal{S} = (\mathcal{S}_1, \ldots, \mathcal{S}_n)$, and $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n) \in (\mathbb{R}^+)^n$, then, with $\mathcal{Q}_i = \operatorname{conv} \mathcal{S}_i$,*

$$
(11) \qquad \operatorname{Vol}_n(\lambda_1 \mathcal{Q}_1 + \cdots + \lambda_n \mathcal{Q}_n) = \sum_{C \in \mathcal{D}} \lambda_1^{k_1^C} \cdots \lambda_n^{k_n^C} \operatorname{Vol}_n(C)
$$

where for each $C \in \mathcal{D}$, $(k_1^C, \ldots, k_n^C)$ denotes the type of C, i.e., $\operatorname{Vol}_n(\lambda_1 \mathcal{Q}_1 + \cdots + \lambda_n \mathcal{Q}_n)$ is a homogeneous polynomial of degree n in $(\lambda_1, \ldots, \lambda_n)$.

Proof. By Proposition 4, $\boldsymbol{\lambda} \circ \mathcal{D} = \{\boldsymbol{\lambda} \circ C \mid C \in \mathcal{D}\}$ is also a fine mixed subdivision of $\boldsymbol{\lambda} \circ \mathcal{S}$. Therefore the volume $\operatorname{Vol}_n(\lambda_1 \mathcal{Q}_1 + \cdots + \lambda_n \mathcal{Q}_n)$ is the sum of the volumes of the cells in $\boldsymbol{\lambda} \circ \mathcal{D}$, and by applying Proposition 11 formula (11) holds.

Also note that for each cell $C \in \mathcal{D}$ of type $(k_1^C, \ldots, k_n^C)$, condition **(d1)** in the definition of fine mixed subdivision requires that $k_1^C + \cdots + k_n^C$ must be exactly n. Therefore each term in the above polynomial expression has a (total) degree of n in $(\lambda_1, \ldots, \lambda_n)$, and hence it is homogeneous. $\qquad \square$

It follows that in the homogeneous polynomial $\operatorname{Vol}_n(\lambda_1 \mathcal{Q}_1 + \cdots + \lambda_n \mathcal{Q}_n)$, the contribution of each cell in $\mathcal{D}$ is a monomial term in $\lambda_1, \ldots, \lambda_n$. Its coefficient is determined by the volume of the cell, and its exponents are given by the type of the cell. Recall that, by definition, the mixed volume $\mathcal{M}(\mathcal{Q}_1, \ldots, \mathcal{Q}_n)$ is the coefficient of $\lambda_1 \times \cdots \times \lambda_n$ in $\operatorname{Vol}_n(\lambda_1 \mathcal{Q}_1 + \cdots + \lambda_n \mathcal{Q}_n)$, so, from (11),

$$
(12) \qquad \mathcal{M}(\mathcal{Q}_1, \ldots, \mathcal{Q}_n) = \sum_{\substack{C \in \mathcal{D} \\ \operatorname{type}(C) = (1, \ldots, 1)}} \operatorname{Vol}_n(C).
$$

Moreover, since $\mathcal{D}$ is a *fine* mixed subdivision, each cell $C \in \mathcal{D}$ of type $(1, \ldots, 1)$ is necessarily of the form $C = (\{\boldsymbol{c}_1, \boldsymbol{c}_1'\}, \ldots, \{\boldsymbol{c}_n, \boldsymbol{c}_n'\})$ where $\{\boldsymbol{c}_i, \boldsymbol{c}_i'\} \subseteq \mathcal{S}_i$. So each $V_i(C)$, defined in (5), consists of a single column, and

$$
V(C) = \begin{bmatrix} \boldsymbol{c}_1 - \boldsymbol{c}_1' & \cdots & \boldsymbol{c}_n - \boldsymbol{c}_n' \end{bmatrix}
$$

as defined in (6). Accordingly, from (10),

$$
(13) \quad \mathcal{M}(\mathcal{Q}_1, \ldots, \mathcal{Q}_n) = \sum_{(\{\boldsymbol{c}_1, \boldsymbol{c}_1'\}, \ldots, \{\boldsymbol{c}_n, \boldsymbol{c}_n'\}) \in \mathcal{D}} \left| \det \begin{bmatrix} \boldsymbol{c}_1 - \boldsymbol{c}_1' & \cdots & \boldsymbol{c}_n - \boldsymbol{c}_n' \end{bmatrix} \right|.
$$

Remark 13. Formula (12) reveals an important property: The mixed volume is indeed nonnegative, which is not immediately obvious from its definition. Moreover, when $\mathcal{S}$ consists of integral lattice points the mixed volume $\mathcal{M}(\mathcal{Q})$ is an integer and in fact counts the generic number of complex roots of a polynomial system as in the BERNSHTEIN's theorem [1, 16, 23, 27].

The above construction suggests a clear strategy for computing mixed volume: With a fine mixed subdivision of $\mathcal{S} = (\mathcal{S}_1, \ldots, \mathcal{S}_n)$, one may systematically enumerate all the mixed cells of type $(1, \ldots, 1)$, and then the sum of the volume of all these cells as given in (10) is precisely the mixed volume $\mathcal{M}(\mathcal{Q}_1, \ldots, \mathcal{Q}_n)$.

5. Mixed subdivisions induced by generic lifting

We now discuss a constructive procedure, developed in [2, 16], with which a fine mixed subdivision of $\mathcal{S} = (\mathcal{S}_1, \ldots, \mathcal{S}_n)$ can be found.

For each $i = 1, \ldots, n$, let $\boldsymbol{\omega}_i : \mathcal{S}_i \to \mathbb{R}$ be a function that assigns each point in $\mathcal{S}_i$ a real number. The function $\boldsymbol{\omega} := (\boldsymbol{\omega}_1, \ldots, \boldsymbol{\omega}_n)$ is known as a **lifting function** on $\mathcal{S} = (\mathcal{S}_1, \ldots, \mathcal{S}_n)$. For each $i = 1, \ldots, n$, consider the "lifted" versions of $\mathcal{S}_i$ and $\operatorname{conv} \mathcal{S}$ given by

$$\hat{\mathcal{S}}_i := \{(\boldsymbol{a}, \boldsymbol{\omega}_i(\boldsymbol{a})) : \boldsymbol{a} \in \mathcal{S}_i\} \qquad \text{and} \qquad \operatorname{conv} \hat{\mathcal{S}} := \hat{\mathcal{S}}_1 + \cdots + \hat{\mathcal{S}}_n$$

respectively. Let $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ be the projection by erasing the last coordinate, so $\pi(\hat{\mathcal{S}}_i) = \mathcal{S}_i$ for each $i = 1, \ldots, n$, and $\pi(\operatorname{conv} \hat{\mathcal{S}}) = \operatorname{conv} \mathcal{S}$.

Consider the polytope $\operatorname{conv} \hat{\mathcal{S}}$, now in $\mathbb{R}^{n+1}$. We are interested in its "lower hull" with respect to the projection π: A vector $\hat{\boldsymbol{\alpha}} \in \mathbb{R}^{n+1}$ is said to be *upward pointing* if it has positive last coordinate. Without loss of generality, we may assume the last coordinate of an upward pointing $\hat{\boldsymbol{\alpha}}$ is 1, that is, $\hat{\boldsymbol{\alpha}} = (\alpha_1, \ldots, \alpha_n, 1) \in \mathbb{R}^{n+1}$. A face $\hat{F}$ of $\operatorname{conv} \hat{\mathcal{S}}$ is called a **lower face** if its inner normal is upward pointing, namely, there exists an $\hat{\boldsymbol{\alpha}} = (\alpha_1, \ldots, \alpha_n, 1) \in \mathbb{R}^{n+1}$ such that $\hat{F} = (\operatorname{conv} \hat{\mathcal{S}})_{\hat{\boldsymbol{\alpha}}}$. It is important to note that for a lower face $\hat{F}$ of $\operatorname{conv} \hat{\mathcal{S}}$, by linearity,

$$\hat{F} = (\operatorname{conv} \hat{\mathcal{S}})_{\hat{\boldsymbol{\alpha}}} = (\operatorname{conv} \hat{\mathcal{S}}_1)_{\hat{\boldsymbol{\alpha}}} + \cdots + (\operatorname{conv} \hat{\mathcal{S}}_n)_{\hat{\boldsymbol{\alpha}}}$$

for some upward pointing normal $\hat{\boldsymbol{\alpha}}$. In other words, a lower face of $\operatorname{conv} \hat{\mathcal{S}}$ is necessarily a Minkowski sum of n lower faces of $\operatorname{conv} \hat{\mathcal{S}}_1, \ldots, \operatorname{conv} \hat{\mathcal{S}}_n$ respectively sharing a common inner normal of the form $\hat{\boldsymbol{\alpha}} = (\alpha_1, \ldots, \alpha_n, 1)$. The **lower hull** of $\operatorname{conv} \hat{\mathcal{S}}$ is the collection of all of its n-dimensional lower faces.

We shall impose a "genericity" condition on the lifting function. To facilitate the discussion, the following notation will be used: Fix any $i \in \{1, \ldots, n\}$ and a subset $X_i = \{\boldsymbol{x}_{i,1}, \ldots, \boldsymbol{x}_{i,m_i}\} \subseteq \mathcal{S}_i$, containing m_i points for some $m_i > 0$, define

$$(14) \quad V(X_i) = \begin{pmatrix} \boldsymbol{x}_{i,2}^\top - \boldsymbol{x}_{i,1}^\top \\ \boldsymbol{x}_{i,3}^\top - \boldsymbol{x}_{i,1}^\top \\ \vdots \\ \boldsymbol{x}_{i,m_i}^\top - \boldsymbol{x}_{i,1}^\top \end{pmatrix} \quad \text{and} \quad \Omega(X_i) = \begin{pmatrix} \boldsymbol{\omega}_i(\boldsymbol{x}_{i,1}) - \boldsymbol{\omega}_i(\boldsymbol{x}_{i,2}) \\ \boldsymbol{\omega}_i(\boldsymbol{x}_{i,1}) - \boldsymbol{\omega}_i(\boldsymbol{x}_{i,2}) \\ \vdots \\ \boldsymbol{\omega}_i(\boldsymbol{x}_{i,1}) - \boldsymbol{\omega}_i(\boldsymbol{x}_{i,m_i}) \end{pmatrix}.$$

Definition 14. A lifting function $\boldsymbol{\omega} = (\boldsymbol{\omega}_1, \ldots, \boldsymbol{\omega}_n)$ for $\mathcal{S} = (\mathcal{S}_1, \ldots, \mathcal{S}_n)$ is said to be **generic** if for any choice of n (possibly empty) subsets $X_i = \{\boldsymbol{x}_{i,1}, \ldots, \boldsymbol{x}_{i,m_i}\} \subseteq \mathcal{S}_i$ for $i = 1, \ldots, n$ with $m_i \geq 0$ the linear system

$$(15) \qquad \begin{pmatrix} V(X_1) \\ V(X_2) \\ \vdots \\ V(X_n) \end{pmatrix} \cdot \boldsymbol{\alpha} = \begin{pmatrix} \Omega(X_1) \\ \Omega(X_2) \\ \vdots \\ \Omega(X_n) \end{pmatrix}$$

in $\boldsymbol{\alpha}$ has an isolated solution only when the rank of the matrix on the left equals the number of its rows. Note that if the subset X_i is empty, the blocks $V(X_i)$ and $\Omega(X_i)$ will not appear in the above equation.

Remark 15. One important observation is that *almost all* liftings are generic in the sense of Definition 14, justifying the choice of the terminology. More precisely, for each $i = 1, \ldots, n$, we can identify $\boldsymbol{\omega}_i : \mathcal{S}_i \to \mathbb{R}$ with its images and consider $\boldsymbol{\omega}_i$ as an element in $\mathbb{R}^{N_i}$ where $N_i = \#\mathcal{S}_i$. Similarly, we consider $\boldsymbol{\omega} = (\boldsymbol{\omega}_1, \ldots, \boldsymbol{\omega}_n)$ as an element in $\mathbb{R}^N$ where $N = N_1 + \cdots + N_n$. If $\boldsymbol{\omega}$ is not generic, then there exists a choice of n (possibly empty) subsets $\{\boldsymbol{x}_{i,1}, \ldots, \boldsymbol{x}_{i,m_i}\} \subseteq \mathcal{S}_i$ for $i = 1, \ldots, n$ with $m_i > 0$ for which the rank of the matrix on the left hand side of the linear system (15) is less than the number of its rows but the system has a solution. This condition forces $\boldsymbol{\omega}$ to be in an affine subspace of lower dimension. Since there are only finitely many ways of choosing subsets of $\mathcal{S}_1, \ldots, \mathcal{S}_n$, the set of non-generic liftings is thus contained in a finite union of lower dimensional affine subspaces of $\mathbb{R}^N$ determined by the points in $\mathcal{S}$. This set is necessarily of measure zero. Indeed, this set is closed and nowhere dense. This is of great practical importance: one can choose a lifting at random, then the probability of choosing a non-generic one is zero.

Proposition 16 (Induced fine mixed subdivision). *Let $\boldsymbol{\omega} = (\boldsymbol{\omega}, \ldots, \boldsymbol{\omega}_n)$ be a generic lifting for $\mathcal{S} = (\mathcal{S}_1, \ldots, \mathcal{S}_n)$, and let $\hat{\mathcal{D}}_{\boldsymbol{\omega}}$ be the collection of all $\hat{C} = (\hat{C}_1, \ldots, \hat{C}_n)$ with $\hat{C}_i \subseteq \hat{\mathcal{S}}_i$ for each $i = 1, \ldots, n$ such that*

1) *conv $\hat{C}_i$ is a face of conv $\hat{\mathcal{S}}_i$ for each $i = 1, \ldots, n$;*

2) *The n faces conv $\hat{C}_i$ of conv $\hat{\mathcal{S}}_1, \ldots,$ conv $\hat{\mathcal{S}}_n$ respectively share a common inner normal of the form $\hat{\boldsymbol{\alpha}} = (\alpha_1, \ldots, \alpha_n, 1)$; and*

3) $\dim(\operatorname{conv}(\hat{C})_1 + \cdots + \operatorname{conv}(\hat{C})_n) = n$.

Then the projection

$$\mathcal{D}_{\boldsymbol{\omega}} = \{(\pi(\hat{C}_1), \ldots, \pi(\hat{C}_n)) \mid \hat{C} = (\hat{C}_1, \ldots, \hat{C}_n) \in \hat{\mathcal{D}}_{\boldsymbol{\omega}}\}$$

*form a fine mixed subdivision of $\mathcal{S} = (\mathcal{S}_1, \ldots, \mathcal{S}_n)$, called the **fine mixed subdivision induced by the lifting function** $\boldsymbol{\omega} = (\boldsymbol{\omega}, \ldots, \boldsymbol{\omega})$.*

Proof. We shall prove $\mathcal{D}_{\boldsymbol{\omega}}$ satisfies the conditions **(a),(b),(c),(d1),(d2)** and **(e)** in Definitions 1, 2, and 3 of a fine mixed subdivision of $\mathcal{S} = (\mathcal{S}_1, \ldots, \mathcal{S}_n)$.

(a) We need to show that for any $C = (C_1, \ldots, C_n) \in \mathcal{D}_{\boldsymbol{\omega}}$, $\dim(\operatorname{conv} C) = n$. Since conv $\hat{C}$ is a lower facet of conv $\hat{\mathcal{S}}$, there exists a vector $\hat{\boldsymbol{\alpha}} = (\boldsymbol{\alpha}, 1) \in \mathbb{R}^{n+1}$ such that conv $\hat{C} = (\operatorname{conv} \hat{\mathcal{S}})_{\hat{\boldsymbol{\alpha}}}$. Now suppose $\dim(\operatorname{conv} C) < n$, then conv C must lie in some hyperplane in $\mathbb{R}^n$, that is, there exists a vector $\boldsymbol{\beta} \in \mathbb{R}^n$ such that conv $C \subseteq \{\boldsymbol{x} \in \mathbb{R}^n : \langle \boldsymbol{x}, \boldsymbol{\beta} \rangle = h\}$ for some $h \in \mathbb{R}$. Then for any $\hat{\boldsymbol{x}} = (x_1, \ldots, x_{n+1}) \in \operatorname{conv} \hat{C}$, $\boldsymbol{x} := \pi((x_1, \ldots, x_{n+1})) \in \operatorname{conv} C$, and so $\langle \hat{\boldsymbol{x}}, (\boldsymbol{\beta}, 0) \rangle = \langle \boldsymbol{x}, \boldsymbol{\beta} \rangle + x_{n+1} \cdot 0 = h$. Thus conv $\hat{C}$ satisfies two independent linear equations in $\mathbb{R}^{n+1}$:

$$\langle \hat{\boldsymbol{x}}, (\boldsymbol{\alpha}, 1) \rangle = \operatorname{ht}_{\hat{\boldsymbol{\alpha}}}(\operatorname{conv} \hat{C})$$
$$\langle \hat{\boldsymbol{x}}, (\boldsymbol{\beta}, 0) \rangle = h,$$

and its dimension is therefore at most $n + 1 - 2 = n - 1$, contradicting to the fact that conv $\hat{C}$ is a facet of an $(n+1)$-dimensional polytope conv $\hat{\mathcal{S}}$ by construction. So conv C must be n-dimensional.

(b) For any two distinct cells $A, B \in \mathcal{D}_{\boldsymbol{\omega}}$, if $F = (\operatorname{conv} A) \cap (\operatorname{conv} B) \subset \mathbb{R}^n$ is nonempty, we need to show that F is a common face of both conv A and conv B. Let $\hat{F} := \operatorname{conv} \hat{A} \cap \operatorname{conv} \hat{B} \subset \mathbb{R}^{n+1}$. Clearly, F is a projection of $\hat{F}$ onto $\{\boldsymbol{e}_{n+1}\}^{\perp}$ where $\boldsymbol{e}_{n+1} = (0, \ldots, 0, 1)^{\top} \in \mathbb{R}^{n+1}$. By definition, conv $\hat{A}$ and conv $\hat{B}$ are lower facets of conv $\hat{\mathcal{S}}$, so there exist vectors $\hat{\boldsymbol{\alpha}} = (\boldsymbol{\alpha}, 1)$ and $\hat{\boldsymbol{\beta}} = (\boldsymbol{\beta}, 1)$ in $\mathbb{R}^{n+1}$ such that conv $\hat{A} = (\operatorname{conv} \hat{\mathcal{S}})_{\hat{\boldsymbol{\alpha}}}$ and conv $\hat{B} = (\operatorname{conv} \hat{\mathcal{S}})_{\hat{\boldsymbol{\beta}}}$. Let $h_1 = \operatorname{ht}_{\hat{\boldsymbol{\alpha}}}(\operatorname{conv} \hat{\mathcal{S}})$ and $h_2 = \operatorname{ht}_{\hat{\boldsymbol{\beta}}}(\operatorname{conv} \hat{\mathcal{S}})$.

For any $\boldsymbol{x} \in F$, there exists $t \in \mathbb{R}$ such that $\hat{\boldsymbol{x}} = (\boldsymbol{x}, t) \in \hat{F} = \text{conv}\,\hat{A}$ $\cap\,\text{conv}\,\hat{B}$. Thus

$$h_1 = \langle \hat{\boldsymbol{x}}, \hat{\boldsymbol{\alpha}} \rangle = \langle \boldsymbol{x}, \boldsymbol{\alpha} \rangle + t, \qquad h_2 = \langle \hat{\boldsymbol{x}}, \hat{\boldsymbol{\beta}} \rangle = \langle \boldsymbol{x}, \boldsymbol{\beta} \rangle + t.$$

Taking the difference of these two equations yields

$$\langle \boldsymbol{x}, \boldsymbol{\beta} - \boldsymbol{\alpha} \rangle = h_2 - h_1,$$

which holds for any $\boldsymbol{x} \in F$. For any $\boldsymbol{x} \in \text{conv}\,A \setminus F$, there exists $s \in \mathbb{R}$ such that $\hat{\boldsymbol{x}} = (\boldsymbol{x}, s)$ is in $\text{conv}\,\hat{A}$ but not in $\text{conv}\,\hat{B}$. Consequently,

$$h_1 = \langle \hat{\boldsymbol{x}}, \hat{\boldsymbol{\alpha}} \rangle = \langle \boldsymbol{x}, \boldsymbol{\alpha} \rangle + s, \qquad h_2 < \langle \hat{\boldsymbol{x}}, \hat{\boldsymbol{\beta}} \rangle = \langle \boldsymbol{x}, \boldsymbol{\beta} \rangle + s.$$

Again, taking the difference of these two equations results in

$$\langle \boldsymbol{x}, \boldsymbol{\beta} - \boldsymbol{\alpha} \rangle > h_2 - h_1.$$

Therefore the linear functional $\langle \bullet, \boldsymbol{\beta} - \boldsymbol{\alpha} \rangle$ minimizes over $\text{conv}\,A$ at F, and hence F is a face of $\text{conv}\,A$. By the same argument, functional $\langle \bullet, \boldsymbol{\alpha} - \boldsymbol{\beta} \rangle$ minimizes over $\text{conv}\,B$ at F, and hence F is a face of $\text{conv}\,B$ as well. Therefore F is a common face of $\text{conv}\,A$ and $\text{conv}\,B$.

(c) We need to show $\bigcup_{C \in \mathcal{D}_{\boldsymbol{\omega}}} \text{conv}\,C = \text{conv}\,\mathcal{S}$. Clearly, $\text{conv}\,C \subseteq \text{conv}\,\mathcal{S}$ for each $C \in \mathcal{D}_{\boldsymbol{\omega}}$, so $\bigcup_{C \in \mathcal{D}_{\boldsymbol{\omega}}} \text{conv}\,C \subseteq \text{conv}\,S$. For the other direction, fix any $\boldsymbol{q} \in \text{conv}\,\mathcal{S}$, let $W = (\{\boldsymbol{q}\} \times \mathbb{R}) \cap \text{conv}\,\hat{\mathcal{S}}$, which is a closed set. Since $\text{conv}\,\hat{\mathcal{S}}$ is a polytope, W must be bounded and hence compact, so $\ell = \min\{t \in \mathbb{R} : (\boldsymbol{q}, t) \in \text{conv}\,\hat{\mathcal{S}}\}$ exists. Let $\hat{\boldsymbol{q}} = (\boldsymbol{q}, \ell)$, then $\hat{\boldsymbol{q}} \in \partial\,(\text{conv}\,\hat{\mathcal{S}})$, so there must be an n-dimensional facet $\hat{F}$ of $\text{conv}\,\hat{\mathcal{S}}$ containing $\hat{\boldsymbol{q}}$. We shall show that $\hat{F}$ is indeed a lower facet.

First, if $\dim(\text{conv}\,\hat{\mathcal{S}}) = n$, then all faces are lower faces. Otherwise, let $\hat{\boldsymbol{\alpha}} = (\alpha_1, \dots, \alpha_{n+1})$ be the inner normal of $\hat{F}$. For $\epsilon > 0$ small enough so that $(\boldsymbol{q}, \ell + \epsilon) \in \text{conv}\,\hat{\mathcal{S}}$, then

$$\langle (\boldsymbol{q}, \ell + \epsilon), \hat{\boldsymbol{\alpha}} \rangle = \langle \hat{\boldsymbol{q}}, \hat{\boldsymbol{\alpha}} \rangle + \epsilon \cdot \alpha_{n+1} > \langle \hat{\boldsymbol{q}}, \hat{\boldsymbol{\alpha}} \rangle$$

since $\langle \bullet, \hat{\boldsymbol{\alpha}} \rangle$ minimizes on $\text{conv}\,\hat{\mathcal{S}}$ at $\hat{F}$. This implies $\alpha_{n+1} > 0$. Therefore $\hat{F}$ is a lower facet, and so there is a cell $C \in \mathcal{D}_{\boldsymbol{\omega}}$ such that $\text{conv}\,\hat{C} = \hat{F}$; in particular, $\boldsymbol{q} \in \text{conv}\,C$. Therefore $\bigcup_{C \in \mathcal{D}_{\boldsymbol{\omega}}} \text{conv}\,C \supseteq \text{conv}\,\mathcal{S}$.

(d2) Let cells $A = (A_1, \ldots, A_n), B = (B_1, \ldots, B_n) \in \mathcal{D}_{\boldsymbol{\omega}}$ with $F := (\operatorname{conv} A)$ $\cap (\operatorname{conv} B)$ being nonempty. We shall show

$$\operatorname{conv} A \cap \operatorname{conv} B = \sum_{i=1}^{n} (\operatorname{conv} A_i) \cap (\operatorname{conv} B_i).$$

One direction of the containment is obvious: since for any collection of $\boldsymbol{x}_i \in (\operatorname{conv} A_i) \cap (\operatorname{conv} B_i)$ for $i = 1, \ldots, n$, $\boldsymbol{x} := \boldsymbol{x}_1 + \cdots + \boldsymbol{x}_n$ is in both $\operatorname{conv} A$ and $\operatorname{conv} B$ and hence $\sum_{i=1}^{n}(\operatorname{conv} A_i) \cap (\operatorname{conv} B_i) \subseteq (\operatorname{conv} A) \cap (\operatorname{conv} B)$.

For the other direction, since $\operatorname{conv} \hat{B}$ is a lower facet of $\operatorname{conv} \hat{\mathcal{S}}$, let $\hat{\boldsymbol{\beta}} = (\boldsymbol{\beta}, 1) \in \mathbb{R}^{n+1}$ be its inner normal, then

$$\operatorname{conv} \hat{B} = (\operatorname{conv} \hat{\mathcal{S}})_{\hat{\boldsymbol{\beta}}} = \sum_{i=1}^{n} (\operatorname{conv} \hat{\mathcal{S}}_i)_{\hat{\boldsymbol{\beta}}} \ \text{ with } \ (\hat{\mathcal{S}}_i)_{\hat{\boldsymbol{\beta}}} = \operatorname{conv} \hat{B}_i.$$

Picking any $\boldsymbol{a} \in F$, consider its corresponding $\hat{\boldsymbol{a}} \in (\operatorname{conv} \hat{A}) \cap (\operatorname{conv} \hat{B})$. Then $\hat{\boldsymbol{a}} = \hat{\boldsymbol{a}}_1 + \cdots + \hat{\boldsymbol{a}}_n$ for some $\hat{\boldsymbol{a}}_i \in \operatorname{conv} \hat{A}_i$. Moreover, $\hat{\boldsymbol{a}} \in \operatorname{conv} \hat{B}$ implies

$$\langle \hat{\boldsymbol{a}}, \hat{\boldsymbol{\beta}} \rangle = \sum_{i=1}^{n} \langle \hat{\boldsymbol{a}}_i, \hat{\boldsymbol{\beta}} \rangle = \operatorname{ht}_{\hat{\boldsymbol{\beta}}}(\operatorname{conv} \hat{\mathcal{S}}) = \sum_{i=1}^{n} \operatorname{ht}_{\hat{\boldsymbol{\beta}}}(\operatorname{conv} \hat{\mathcal{S}}_i).$$

But for each $i = 1, \ldots, n$, $\langle \hat{\boldsymbol{a}}_i, \hat{\boldsymbol{\beta}} \rangle \geq \operatorname{ht}_{\hat{\boldsymbol{\beta}}}(\operatorname{conv} \hat{\mathcal{S}}_i)$. Therefore $\langle \hat{\boldsymbol{a}}_i, \hat{\boldsymbol{\beta}} \rangle = \operatorname{ht}_{\hat{\boldsymbol{\beta}}}(\operatorname{conv} \hat{\mathcal{S}}_i)$ for $i = 1, \ldots, n$, and hence each $\hat{\boldsymbol{a}}_i \in (\operatorname{conv} \hat{\mathcal{S}}_i)_{\hat{\boldsymbol{\beta}}} = \operatorname{conv} \hat{B}_i$. Consequently, $\hat{\boldsymbol{a}}_i \in (\operatorname{conv} \hat{A}_i) \cap (\operatorname{conv} \hat{B}_i)$ for each $i = 1, \ldots, n$, thus $\hat{\boldsymbol{a}} = \sum_{i=1}^{n} \hat{\boldsymbol{a}}_i \in \sum_{i=1}^{n}(\operatorname{conv} \hat{A}_i) \cap (\operatorname{conv} \hat{B}_i)$. Projection via π yields $\boldsymbol{a} \in \sum_{i=1}^{n}(\operatorname{conv} A_i) \cap (\operatorname{conv} B_i)$ and therefore

$$(\operatorname{conv} A) \cap (\operatorname{conv} B) \subseteq \sum_{i=1}^{n} (\operatorname{conv} A_i) \cap (\operatorname{conv} B_i).$$

(e) For a cell $C = (C_1, \ldots, C_n) \in \mathcal{D}_{\boldsymbol{\omega}}$, we fix an i in $\{1, \ldots, n\}$ and consider C_i only. Let $m_i = \#C_i - 1$, then $C_i = \{\boldsymbol{c}_{i,0}, \ldots, \boldsymbol{c}_{i,m_i}\} \subset \mathcal{S}_i$. Let $\hat{\boldsymbol{\alpha}} = (\boldsymbol{\alpha}, 1)$ be an inner normal of $\operatorname{conv} \hat{C}$ in $\operatorname{conv} \hat{\mathcal{S}}$. Then

$$(16) \quad \langle \hat{\boldsymbol{c}}_{i,j}, \hat{\boldsymbol{\alpha}} \rangle = \langle (\boldsymbol{c}_{i,j}, \boldsymbol{\omega}_i(\boldsymbol{c}_{i,j})), (\boldsymbol{\alpha}, 1) \rangle = \langle \boldsymbol{c}_{i,j}, \boldsymbol{\alpha} \rangle + \boldsymbol{\omega}_i(\boldsymbol{c}_{i,j}) = \operatorname{ht}_{\hat{\boldsymbol{\alpha}}}(\operatorname{conv} \hat{\mathcal{S}}_i)$$

for each $j = 0, \ldots, m_i$. This can be written as

$$(17) \qquad \begin{pmatrix} \boldsymbol{c}_{i,1}^{\top} - \boldsymbol{c}_{i,0}^{\top} \\ \vdots \\ \boldsymbol{c}_{i,m_i}^{\top} - \boldsymbol{c}_{i,0}^{\top} \end{pmatrix} \cdot \boldsymbol{\alpha} = \begin{pmatrix} \boldsymbol{\omega}_i(\boldsymbol{c}_{i,0}) - \boldsymbol{\omega}_i(\boldsymbol{c}_{i,1}) \\ \vdots \\ \boldsymbol{\omega}_i(\boldsymbol{c}_{i,0}) - \boldsymbol{\omega}_i(\boldsymbol{c}_{i,m_i}) \end{pmatrix}.$$

Suppose $\operatorname{conv} C_i$ is not a simplex, then those m_i points $\boldsymbol{c}_{i,0}, \ldots, \boldsymbol{c}_{i,m_i}$ must be affinely dependent. It follows that matrix on the left hand side of the above equation is of rank less than m_i. This contradicts to the genericity of the lifting (Definition 14).

(d1) Fix a cell $C = (C_1, \ldots, C_n) \in \mathcal{D}_{\boldsymbol{\omega}}$ with $C_i = \{\boldsymbol{c}_0, \ldots, \boldsymbol{c}_{m_i}\}$ where $m_i = \#C_i - 1$ for each $i = 1, \ldots, n$. Assume condition **(e)** is already satisfied, that is, for each $i = 1, \ldots, n$, $\operatorname{conv} C_i$ is a simplex of dimension m_i. Then it is enough to show that $m_1 + \cdots + m_n = n$. Let $\hat{\boldsymbol{\alpha}} = (\boldsymbol{\alpha}, 1)$ be an inner normal of $\operatorname{conv} \hat{C}$ in $\operatorname{conv} \hat{\mathcal{S}}$. Then $\hat{\boldsymbol{\alpha}}$ satisfies (16) for each $i = 1, \ldots, n$ and $j = 0, \ldots, m_i$. Similar to (17), these equations can be combined into

$$\begin{pmatrix} \boldsymbol{c}_{1,1}^{\top} - \boldsymbol{c}_{1,0}^{\top} \\ \vdots \\ \boldsymbol{c}_{1,m_1}^{\top} - \boldsymbol{c}_{1,0}^{\top} \\ \vdots \\ \boldsymbol{c}_{n,1}^{\top} - \boldsymbol{c}_{n,0}^{\top} \\ \vdots \\ \boldsymbol{c}_{n,m_1}^{\top} - \boldsymbol{c}_{n,0}^{\top} \end{pmatrix} \cdot \boldsymbol{\alpha} = \begin{pmatrix} \boldsymbol{\omega}_1(\boldsymbol{c}_{1,0}) - \boldsymbol{\omega}_1(\boldsymbol{c}_{1,1}) \\ \vdots \\ \boldsymbol{\omega}_1(\boldsymbol{c}_{1,0}) - \boldsymbol{\omega}_1(\boldsymbol{c}_{1,m_1}) \\ \vdots \\ \boldsymbol{\omega}_n(\boldsymbol{c}_{n,0}) - \boldsymbol{\omega}_n(\boldsymbol{c}_{n,1}) \\ \vdots \\ \boldsymbol{\omega}_n(\boldsymbol{c}_{n,0}) - \boldsymbol{\omega}_n(\boldsymbol{c}_{n,m_i}) \end{pmatrix}.$$

Here, the matrix on the left hand side of the above equation is of the size $(m_1 + \cdots + m_n) \times n$. By the genericity assumption (Definition 14) the rank of this matrix must be $m_1 + \cdots + m_n$. Therefore $m_1 + \cdots + m_n \leq n$.

Recall that condition **(a)** asserts $\dim(\operatorname{conv} C) = n$. Since $m_1 + \cdots + m_n = \dim(\operatorname{conv} C_1) + \cdots + \dim(\operatorname{conv} C_n) \geq n$, thus $m_1 + \cdots + m_n = n$.

Therefore $\mathcal{D}_{\boldsymbol{\omega}}$ satisfies condition **(a)-(e)** in Definition 1, 2, and 3, and hence it is a fine mixed subdivision for $\mathcal{S}$. $\square$

6. The semi-mixed case

The n-tuple $\mathcal{S} = (\mathcal{S}_1, \ldots, \mathcal{S}_n)$ is called **semi-mixed** of type $(r_1, \ldots, r_m)$ when $\mathcal{S}_i$'s are not all distinct, but they are equal within m blocks of sizes $r_1, \ldots, r_m$, i.e., there are m sets $\mathcal{S}^{(1)}, \ldots, \mathcal{S}^{(r)} \subset \mathbb{R}^n$ such that $\mathcal{S}^{(i)} = \mathcal{S}_{i,1} = \cdots = \mathcal{S}_{i,r_i}$ where

$$\mathcal{S}_{i,j} \in \{\mathcal{S}_1, \cdots, \mathcal{S}_n\} \qquad \text{for} \quad 1 \le i \le m, \quad 1 \le j \le r_i,$$

and $r_1 + \cdots + r_m = n$. $\mathcal{S}$ is called **unmixed** if $m = 1$, i.e., when all the $\mathcal{S}_i's$ are identical. $\mathcal{S}$ is called **fully mixed** if $m = n$, i.e., when the $\mathcal{S}_i's$ are distinct. We abbreviate $\mathcal{S} = (\mathcal{S}^{(1)}, r_1; \mathcal{S}^{(2)}, r_2; \cdots; \mathcal{S}^{(m)}, r_m)$, and $\mathcal{Q} = (\mathcal{Q}^{(1)}, r_1; \mathcal{Q}^{(2)}, r_2; \cdots; \mathcal{Q}^{(m)}, r_m)$, with $\mathcal{Q}^{(i)} = \text{conv} \, \mathcal{S}^{(i)}$ for $i = 1, \ldots, m$.

To calculate the mixed volume of a semi-mixed system $\mathcal{S} = (\mathcal{S}^{(1)}, r_1; \mathcal{S}^{(2)}, r_2; \cdots; \mathcal{S}^{(m)}, r_m)$, one may, of course, follow the standard procedure described in §4 without paying a special attention to its semi-mixed structure. However, when this special structure is taken into account, a revised procedure may be developed with a great reduction in the amount of computation, especially when $\mathcal{S}$ is unmixed such as the nine-point path synthesis problem for four-bar linkages [30] in mechanical design.

Now, the mixed volume $\mathcal{M}(\mathcal{Q}^{(1)}, r_1; \mathcal{Q}^{(2)}, r_2; \cdots; \mathcal{Q}^{(m)}, r_m)$ is, by definition, the coefficient of $\prod_{j=1}^{r_1} \lambda_{1,j} \cdots \prod_{j=1}^{r_m} \lambda_{m,j}$ in the expansion of the homogeneous polynomial

$$\text{Vol}_n \left(\sum_{i=1}^{m} \sum_{j=1}^{r_i} \lambda_{ij} \mathcal{Q}^{(i)} \right) = \text{Vol}_n \left(\sum_{i=1}^{m} \left(\sum_{j=1}^{r_i} \lambda_{ij} \right) \mathcal{Q}^{(i)} \right).$$

Let

$$\beta_i = \sum_{j=1}^{r_i} \lambda_{i,j} \; \text{ for } i = 1, \ldots, m,$$

then the above expression becomes

$$R(\beta_1, \ldots, \beta_m) := \text{Vol}_n(\beta_1 \mathcal{Q}^{(1)} + \beta_2 \mathcal{Q}^{(2)} + \cdots + \beta_m \mathcal{Q}^{(m)})$$

which is a homogeneous polynomial of degree n in the β_i's. Notice that by the multinomial expansion,

$$\beta_i^k = \left(\sum_{j=1}^{r_i} \lambda_{ij} \right)^k = \sum_{t_1 + \cdots + t_{r_i} = k} \frac{k!}{t_1! \cdots t_{r_i}!} \prod_{j=1}^{r_i} \lambda_{ij}^{t_j}.$$

Therefore a monomial in β_i's can be expanded as

$$\beta_1^{k_1} \cdots \beta_m^{k_m} = \prod_{i=1}^{m} \left(\sum_{t_{i1}+\cdots+t_{ir_i}=k_i} \frac{k_i!}{t_{i1}! \cdots t_{ir_i}!} \prod_{j=1}^{r_i} \lambda_{ij}^{t_{ij}} \right).$$

Clearly, such an expansion involves the monomial $\prod_{i=1}^{m} \prod_{j=1}^{r_1} \lambda_{1,j}$ precisely when $t_{i,j} = 1$ for all $i = 1, \ldots, m$ and $j = 1, \ldots, r_i$ which yields $k_i = \sum_{j=1}^{r_i} t_{i,j} = \sum_{j=1}^{r_i} 1 = r_i$. Therefore in

$$R(\beta_1, \ldots, \beta_m) = R\left(\left(\sum_{j=1}^{r_1} \lambda_{1,j} \right), \ldots, \left(\sum_{j=1}^{r_m} \lambda_{m,j} \right) \right),$$

only the monomial $\beta_1^{r_1} \cdots \beta_m^{r_m}$ involves $\prod_{i=1}^{m} \prod_{j=1}^{r_1} \lambda_{i,j}$ in which it appears with coefficient $r_1! \cdots r_m!$.

Proposition 17. *For semi-mixed system $\mathcal{S} = (\mathcal{S}^{(1)}, r_1; \mathcal{S}^{(2)}, r_2; \cdots; \mathcal{S}^{(m)}, r_m)$, the mixed volume $\mathcal{M}(\mathcal{Q}^{(1)}, r_1; \mathcal{Q}^{(2)}, r_2; \cdots; \mathcal{Q}^{(m)}, r_m)$ is the coefficient of $\beta_1^{r_1} \cdots \beta_m^{r_m}$ in the polynomial*

$$\mathrm{Vol}_n(\beta_1 \mathcal{Q}^{(1)} + \beta_2 \mathcal{Q}^{(2)} + \cdots + \beta_m \mathcal{Q}^{(m)})$$

divided by $r_1! \cdots r_m!$.

To utilize this observation in mixed volume computations, we slightly extend the framework of *cells* in which the mixed subdivision is defined for the Minkowski sum of polytopes in previous sections. With $\mathcal{S}^{(i)}$ being a finite subset of $\mathbb{R}^n$ for $i = 1, \ldots, m$, and $m \leq n$, a **cell** of the m-tuple $\mathcal{S} = (\mathcal{S}^{(1)}, \ldots, \mathcal{S}^{(m)})$ is now an m-tuple $C = (C_1, \ldots, C_m)$ of nonempty subsets $C_i \subseteq \mathcal{S}^{(i)}$. With similar notations

$$\mathrm{type}\, C = (\dim(\mathrm{conv}\, C_1), \ldots, \dim(\mathrm{conv}\, C_m))$$
$$\mathrm{conv}\, C = \mathrm{conv}\, C_1 + \cdots + \mathrm{conv}\, C_m$$
$$\mathrm{Vol}_n\, C = \mathrm{Vol}_n(\mathrm{conv}\, C),$$

we give the following:

Definition 18 (Fine semi-mixed subdivision). A **fine semi-mixed subdivision** of the m-tuple $\mathcal{S} = (\mathcal{S}^{(1)}, \ldots, \mathcal{S}^{(m)})$ is a collection $\mathcal{D}$ of cells of $\mathcal{S}$ such that

(a): $\dim(\operatorname{conv} C) = n$ for all $C \in \mathcal{D}$;

(b): For a distinct pair $A, B \in \mathcal{D}$, if $(\operatorname{conv} A) \cap (\operatorname{conv} B)$ is nonempty, then it is a common face of both;

(c): $\bigcup_{C \in \mathcal{D}} \operatorname{conv} C = \operatorname{conv} \mathcal{S}^{(1)} + \cdots + \operatorname{conv} \mathcal{S}^{(m)}$;

(d1): For each cell $C = (C_1, \ldots, C_m) \in \mathcal{D}$, $\sum_{i=1}^{m} \dim(\operatorname{conv} C_i) = n$

(d2): For distinct pair of cells $A = (A_1, \ldots, A_m), B = (B_1, \ldots, B_m) \in \mathcal{D}$,

$$(\operatorname{conv} A) \cap (\operatorname{conv} B) = \sum_{i=1}^{m} (\operatorname{conv} A_i \cap \operatorname{conv} B_i);$$

(e): For each cell $C = (C_1, \ldots, C_m) \in \mathcal{D}$, $\operatorname{conv} C_i$ is a simplex of dimension $\#C_i - 1$ for $i = 1, \ldots, m$.

Notice that replacing all the m's by n in the above yields exactly the same *fine mixed subdivision* in Definition 1, 2, and 3. Most importantly, the properties of fine mixed subdivisions proved in §2 can be preserved with minor adjustments. In particular, the scaling invariance of a fine semi-mixed subdivision remains valid:

Proposition 19. *If $\mathcal{D}$ is a fine semi-mixed subdivision of the m-tuple $\mathcal{S} = (\mathcal{S}^{(1)}, \ldots, \mathcal{S}^{(m)})$, and $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_m) \in (\mathbb{R}^+)^m$, the set*

$$\boldsymbol{\beta} \circ \mathcal{D} := \{\boldsymbol{\beta} \circ C = (\beta_1 C_1, \ldots, \beta_m C_m) \; : \; C = (C_1, \ldots, C_m) \in \mathcal{D}\}$$

forms a fine semi-mixed subdivision of $\boldsymbol{\beta} \circ \mathcal{S} = (\beta_1 \mathcal{S}^{(1)}, \ldots, \beta_m \mathcal{S}^{(m)})$.

Also, similar to (11), the volume $\operatorname{Vol}_n(\beta_1 \mathcal{Q}^{(1)} + \beta_2 \mathcal{Q}^{(2)} + \cdots + \beta_m \mathcal{Q}^{(m)})$ can be expressed in terms of volumes of cells of a fine semi-mixed subdivision:

Proposition 20. *If $\mathcal{D}$ is a fine semi-mixed subdivision of the m-tuple $\mathcal{S} = (\mathcal{S}^{(1)}, \ldots, \mathcal{S}^{(m)})$, and $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_m) \in (\mathbb{R}^+)^m$,*

$$(18) \qquad \operatorname{Vol}_n(\beta_1 \mathcal{Q}^{(1)} + \cdots + \beta_m \mathcal{Q}^{(m)}) = \sum_{C \in \mathcal{D}} \beta_1^{k_1^C} \cdots \beta_m^{k_m^C} \operatorname{Vol}_n(C)$$

where $(k_1^C, \ldots, k_m^C)$ denotes the type of $C = (C_1, \ldots, C_m) \in \mathcal{D}$.

Similar to (6), in this semi-mixed context we define

$$(19) \qquad V(C) = \begin{bmatrix} V(C_1) & \cdots & V(C_m) \end{bmatrix}$$

in which the block $V(C_i)$ will not appear if $\#C_i = 1$. When $\mathcal{D}$ is a fine semi-mixed subdivision, similar to (10), the volume of a cell $C = (C_1, \ldots, C_m) \in \mathcal{D}$ can be computed via the formula

$$\mathrm{Vol}_n(C) = \frac{1}{k_1^C! \cdots k_m^C!} \, |\det V(C)|$$

where $(k_1^C, \ldots, k_m^C) = \mathrm{type}(C)$. This yields the expression of the mixed volume of a semi-mixed system as a generalization of (12):

Proposition 21. *Let $\mathcal{D}$ be a fine semi-mixed subdivision of the semi-mixed system $\mathcal{S} = (\mathcal{S}^{(1)}, r_1; \cdots ; \mathcal{S}^{(m)}, r_m)$. The mixed volume $\mathcal{M}(\mathcal{Q}^{(1)}, r_1; \cdots ; \mathcal{Q}^{(m)}, r_m)$ of this system is*

$$(20) \qquad \mathcal{M}(\mathcal{Q}^{(1)}, r_1; \mathcal{Q}^{(2)}, r_2; \cdots ; \mathcal{Q}^{(m)}, r_m) = \sum_{\substack{C \in \mathcal{D} \\ \mathrm{type}(C) = (r_1, \ldots, r_m)}} |\det V(C)|,$$

with $V(C)$ as defined in (19).

Moreover, similar to inducing a fine mixed subdivision for the fully mixed case $\mathcal{S} = (\mathcal{S}_1, \ldots, \mathcal{S}_n)$ by a generic lifting function as described in §5, the same procedure can be followed almost line by line to construct a induced fine semi-mixed subdivision for a semi-mixed system $\mathcal{S} = (\mathcal{S}^{(1)}, r_1; \cdots ; \mathcal{S}^{(m)}, r_m)$. Hence Proposition 16 can be generalized to the semi-mixed system:

Proposition 22 (Induced fine semi-mixed subdivision). *Let $\boldsymbol{\omega} = (\boldsymbol{\omega}_1, \ldots, \boldsymbol{\omega}_m)$ be a generic lifting function for the m-tuple $\mathcal{S} = (\mathcal{S}^{(1)}, \ldots, \mathcal{S}^{(m)})$. Define*

$$\hat{\mathcal{S}}^{(i)} := \{(\boldsymbol{a}, \boldsymbol{\omega}_i(\boldsymbol{a})) : \boldsymbol{a} \in \mathcal{S}^{(i)}\},$$

for each $i = 1, \ldots, m$. Let $\hat{\mathcal{D}}_{\boldsymbol{\omega}}$ be the collection of all $\hat{C} = (\hat{C}_1, \ldots, \hat{C}_m)$ with $\hat{C}_i \subseteq \hat{\mathcal{S}}^{(i)}$ for each $i = 1, \ldots, m$ such that

1) $\mathrm{conv}\,\hat{C}_i$ *is a face of $\mathrm{conv}\,\hat{\mathcal{S}}_i$ for each $i = 1, \ldots, m$;*

2) *The m faces $\mathrm{conv}\,\hat{C}_i$ of $\mathrm{conv}\,\hat{\mathcal{S}}^{(i)}$ for $i = 1, \ldots, m$ respectively share a common inner normal of the form $\hat{\boldsymbol{\alpha}} = (\alpha_1, \ldots, \alpha_n, 1)$; and*

3) $\dim(\mathrm{conv}\,\hat{C}_1 + \cdots + \mathrm{conv}\,\hat{C}_m) = n$.

Then the projection of $\hat{\mathcal{D}}_{\boldsymbol{\omega}}$

$$\mathcal{D}_{\boldsymbol{\omega}} = \{(\pi(\hat{C}_1), \ldots, \pi(\hat{C}_m))) \, : \, (\hat{C}_1, \ldots, \hat{C}_m) \in \hat{\mathcal{D}}_{\boldsymbol{\omega}}\}$$

*form a fine semi-mixed subdivision of $\mathcal{S} = (\mathcal{S}^{(1)}, \ldots, \mathcal{S}^{(m)})$, and it is called the **fine semi-mixed subdivision induced by the lifting function $\boldsymbol{\omega}$**.*

7. Enumeration of mixed cells

The crux of our strategy for calculating mixed volume $\mathcal{M}(\mathcal{Q}^{(1)}, r_1; \cdots; \mathcal{Q}^{(m)}, r_m)$ of a semi-mixed system $\mathcal{S} = (\mathcal{S}^{(1)}, r_1; \cdots; \mathcal{S}^{(m)}, r_m)$, where $\mathcal{Q}^{(i)} = \text{conv}\,\mathcal{S}^{(i)}$, is the identification of the mixed cells of type $(r_1, \ldots, r_m)$ in the fine semi-mixed subdivision $\mathcal{D}_{\boldsymbol{\omega}}$ induced by the generic lifting function $\boldsymbol{\omega} := (\boldsymbol{\omega}_1, \ldots, \boldsymbol{\omega}_m)$ as provided by Proposition 22. Algebraically, a mixed cell $C \in \mathcal{D}_{\boldsymbol{\omega}}$ of type $(r_1, \ldots, r_m)$, is an m-tuple

$$(21) \qquad C \; = \; (\, \{\boldsymbol{a}_0^{(1)}, \ldots, \boldsymbol{a}_{r_1}^{(1)}\}, \; \ldots \;, \; \{\boldsymbol{a}_0^{(m)}, \ldots, \boldsymbol{a}_{r_m}^{(m)}\} \,)$$

with $\boldsymbol{a}_j^{(i)} \in \mathcal{S}^{(i)}$ for $i = 1, \ldots, m$ and $j = 0, \ldots, r_i$ for which there exists an $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$ such that for each $i = 1, \ldots, m$,

$$(22) \qquad \begin{aligned} \langle \hat{\boldsymbol{a}}_0^{(i)}, \hat{\boldsymbol{\alpha}} \rangle &= \langle \hat{\boldsymbol{a}}_j^{(i)}, \hat{\boldsymbol{\alpha}} \rangle \quad \text{for } j = 0, \ldots, r_i \\ \langle \hat{\boldsymbol{a}}_0^{(i)}, \hat{\boldsymbol{\alpha}} \rangle &\leq \langle \hat{\boldsymbol{a}}, \hat{\boldsymbol{\alpha}} \rangle \quad \text{for } \boldsymbol{a} \in \mathcal{S}^{(i)} \end{aligned}$$

where $\hat{\boldsymbol{\alpha}} := (\boldsymbol{\alpha}, 1)^{\top} = (\alpha_1, \ldots, \alpha_n, 1)^{\top}$. This algebraic description of mixed cells in $\mathcal{D}_{\boldsymbol{\omega}}$ of type $(r_1, \ldots, r_m)$ is the basis on which mixed cell enumeration algorithms are developed: the problem is now reduced to a search problem for choices of $r_1, \ldots, r_m$ points from $\mathcal{S}^{(1)}, \ldots, \mathcal{S}^{(m)}$ respectively for which the system of inequalities (22) has a solution. There are several approaches to carrying out such search. One of the most efficient class of algorithms is based on the idea of systematic "extension of subfaces".

This scheme first locates the r_1-dimensional *lower faces* of $\hat{\mathcal{Q}}^{(1)}$, that is, the faces of $\hat{\mathcal{Q}}_1$ that has an inner normal with 1 being the last coordinate. These faces are known as level-1 "subfaces". Namely, one first finds all possible choices $\{\boldsymbol{a}_0^{(1)}, \ldots, \boldsymbol{a}_{r_1}^{(1)}\}$ of $r_1 + 1$ points in $\mathcal{S}^{(1)}$ for which there exists an $\hat{\boldsymbol{\alpha}} = (\alpha_1, \ldots, \alpha_n, 1)^{\top} \in \mathbb{R}^{n+1}$ such that

$$(23) \qquad \begin{aligned} \langle \hat{\boldsymbol{a}}_0^{(1)}, \hat{\boldsymbol{\alpha}} \rangle &= \langle \hat{\boldsymbol{a}}_j^{(1)}, \hat{\boldsymbol{\alpha}} \rangle \quad \text{for } j = 0, \ldots, r_1 \\ \langle \hat{\boldsymbol{a}}_0^{(i)}, \hat{\boldsymbol{\alpha}} \rangle &\leq \langle \hat{\boldsymbol{a}}, \hat{\boldsymbol{\alpha}} \rangle \quad \text{for } \boldsymbol{a} \in \mathcal{S}^{(1)} \,. \end{aligned}$$

The testing for the existence of solutions for the above system of inequalities is generally known as the "*Phase I*", or *feasibility*, problem in *Linear Programming*.

Then for each of these level-1 subfaces, systematic attempts are made to "extend" it by finding a r_2-dimensional lower face of $\hat{\mathcal{Q}}^{(2)}$ so that the two lower faces (of $\hat{\mathcal{Q}}^{1)}$ and $\hat{\mathcal{Q}}^{(2)}$ respectively) can share a common inner normal of the form $\hat{\boldsymbol{\alpha}} = (\alpha_1, \ldots, \alpha_n, 1)^\top$ of both $\hat{\mathcal{Q}}^{(1)}$ and $\hat{\mathcal{Q}}^{(2)}$. That is, fixing a level-1 subface defined by $\boldsymbol{a}_0^{(1)}, \ldots, \boldsymbol{a}_{r_1}^{(1)}$ one attempts to find a r_2-dimensional lower face $\mathrm{conv}\{\hat{\boldsymbol{a}}_0^{(2)}, \ldots, \hat{\boldsymbol{a}}_{r_2}^{(2)}\}$ of $\hat{\mathcal{Q}}^{(2)}$ with $\hat{\boldsymbol{a}}_j^{(2)} \in \hat{\mathcal{S}}^{(2)}$ for each $j = 0, \ldots, r_2$ such that the system (23) and the system

$$
(24) \qquad
\begin{aligned}
\langle \hat{\boldsymbol{a}}_0^{(2)}, \hat{\boldsymbol{\alpha}} \rangle &= \langle \hat{\boldsymbol{a}}_j^{(2)}, \hat{\boldsymbol{\alpha}} \rangle \quad \text{for } j = 0, \ldots, r_2 \\
\langle \hat{\boldsymbol{a}}_0^{(i)}, \hat{\boldsymbol{\alpha}} \rangle &\leq \langle \hat{\boldsymbol{a}}, \hat{\boldsymbol{\alpha}} \rangle \qquad \text{for } \boldsymbol{a} \in \mathcal{S}^{(2)}
\end{aligned}
$$

have a common solution $\hat{\boldsymbol{\alpha}} = (\alpha_1, \ldots, \alpha_n, 1)^\top \in \mathbb{R}^{n+1}$. The testing for the existence of a common solution is, again, a classical Phase I problem in Linear Programming. Note that this enlarged system of (23) and (24) may not have a solution for all choices of $\{\hat{\boldsymbol{a}}_0^{(2)}, \ldots, \hat{\boldsymbol{a}}_{r_2}^{(2)}\} \subseteq \hat{\mathcal{S}}^{(2)}$. In this case, no lower face of $\hat{\mathcal{Q}}^{(2)}$ can extend the chosen level-1 subface, and the extension attempt will be restarted on some other level-1 subfaces. On the other hand, the possible r_2-dimensional lower faces of $\hat{\mathcal{Q}}^{(2)}$ that can extend the chosen level-1 subface may not be unique, and each of such possible extension is known as a level-2 subface. They will be extended further individually into possible level-3 subfaces. This process may continue until one reaches level-m subfaces, each of which is an m-tuple of lower faces of $\hat{\mathcal{Q}}^{(1)}, \ldots, \hat{\mathcal{Q}}^{(m)}$ with dimensions $r_1, \ldots, r_m$ respectively that share a common inner normal vector of the form $\hat{\boldsymbol{\alpha}} = (\alpha_1, \ldots, \alpha_n, 1)^\top$. This is precisely a cell in $\hat{\mathcal{D}}_{\boldsymbol{\omega}}$ whose projection (by erasing the last coordinate) is a fine semi-mixed cell of type $(r_1, \ldots, r_m)$ in $\mathcal{D}_{\boldsymbol{\omega}}$ as given in Proposition 22. Once all such mixed cells of type $(r_1, \ldots, r_m)$ have been found, the mixed volume $\mathcal{M}(\mathcal{Q}^{(1)}, r_1; \cdots; \mathcal{Q}^{(m)}, r_m)$ of a semi-mixed system $\mathcal{S} = (\mathcal{S}^{(1)}, r_1; \cdots; \mathcal{S}^{(m)}, r_m)$ can be computed via formula (20).

This scheme of "extension of subfaces" has been developed into a class of efficient, robust, and highly parallel algorithms which have been implemented in software packages such as [4, 11, 12, 19, 22, 26].

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