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Counting Equilibria of the Kuramoto Model Using Birationally Invariant Intersection Index*

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Abstract. Synchronization in networks of interconnected oscillators is a fascinating phenomenon that appears naturally in many independent fields of science and engineering. A substantial amount of work has been devoted to understanding all possible synchronization configurations on a given network. In this setting, a key problem is to determine the total number of such configurations. Through an algebraic formulation for tree and cycle graphs, we provide upper bounds on this number using the theory of the birationally invariant intersection index of a family of rational functions. These bounds are significant and make asymptotic improvements over the best existing bound.

Key words. Kuramoto model, birationally invariant intersection index, BKK bound

AMS subject classifications. 14Q99, 52B20, 65H10

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1. Introduction.

The root counting problem for systems of nonlinear equations is fundamental in mathematics and has a wide range of applications. In this paper, we focus on the root counting problem for the *Kuramoto equations* [31], which are a family of nonlinear systems that characterizes a form of "synchronization" conditions in a network of oscillators. Such a network-induced system is naturally equipped with rich algebraic and combinatorial structures. Rooted in the study of spontaneous synchronization in networks of connected oscillators, this ubiquitous nonlinear system has a wide range of applications in physics, biology, chemistry, and engineering [19, 32].

The root counting problem for Kuramoto systems has a long history. In 1982, Baillieul and Byrnes established an upper bound on the number of roots that depends only on the number of oscillators in the network which is given by $\binom{2N-2}{N-1}$ [2], with N being the number of oscillators in the network. Recent studies [13, 36], however, suggest that much tighter upper bounds that depend on network topology may exist. Leveraging the algebraic and combinatorial structure of these systems, we confirm this conjecture for two important classes of network topologies: trees and cycles. In particular, we will demonstrate that this root counting problem is equivalent to the problem of computing the *birationally invariant intersection index* [29] for a certain family of rational functions.

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With this result, and other tools from algebraic geometry and combinatorial geometry, the primary contribution of this paper is establishing the upper bounds of the root counts for Kuramoto systems induced by tree and cycle networks of N oscillators which are 2^{N-1} and $N\binom{N-1}{\lfloor (N-1)/2 \rfloor}$, respectively. We also show that both bounds are sharp when complex roots are considered. These are dramatic improvements over the best known bound $\binom{2N-2}{N-1}$ as the ratio between the new bounds and $\binom{2N-2}{N-1}$ goes to zero as $N \to \infty$. The new bounds also confirm the crucial role of network topology in the exhaustive study of Kuramoto systems. From a computational viewpoint, these explicit sharp upper bounds on the number of roots are also of great importance in numerical methods for finding all roots to Kuramoto equations because they provide explicit stopping criteria for iterative solvers, such as Newton-based solvers, as well as for the homotopy-based monodromy method [21].

The secondary contribution is the general approach of computing the birationally invariant intersection index by finding the appropriate relaxation: By using the much simpler construction of "adjacency polytope bound," the problem is transformed into one of computing normalized volumes for certain polytopes.

The rest of the paper is structured as follows. In section 2, we state the different formulations of the root counting problem that this paper focuses on. In section 3, we briefly review the Kuramoto model and existing results on the number of possible equilibria. Section 4 reviews notation and well-known theorems to be used. In sections 5 and 6, we establish the root count for Kuramoto systems induced by trees and cycles, respectively. Finally, we conclude in section 7.

2. Problem statement.

Kuramoto equations describe the synchronization conditions for the Kuramoto model. This is one of the fundamental mathematical problems in the study of spontaneous synchronization in networks of connected oscillators.

Mathematically, a network of $N = n + 1$ oscillators can be described by a weighted graph $G = (V, E, A)$ in which vertices $V = \{0, \ldots, n\}$ represent the oscillators, edges E represent their connections, and weights $A = \{a_{ij}\}$ represent the *coupling strength* along edges. In isolation, the oscillators have their own natural frequencies $\omega_0, \ldots, \omega_n$. However, in a network of oscillators the tug of war between the oscillators' tendency to oscillate in their own natural frequencies and the influence of their neighbors gives rise to rich and complicated phenomena. This is captured by the Kuramoto model [31]

$$\frac{d\theta_i}{dt} = \omega_i - \sum_{j \in \mathcal{N}_G(i)} a_{ij} \sin(\theta_i - \theta_j) \quad \text{for } i = 0, \ldots, n, \tag{1}$$

where each $\theta_i \in [0, 2\pi)$ is the phase angle that describes the status of the ith oscillator, and $\mathcal{N}_G(i)$ is the set of neighbors of the ith vertex. "Synchronization" occurs when these two forces reach a certain form of equilibrium for all oscillators.

A configuration $\boldsymbol{\theta} = (\theta_0, \ldots, \theta_n)$ is said to be in *frequency synchronization* if $\frac{d\theta_i}{dt} = 0$ for all i at $\boldsymbol{\theta}$. To remove the inherent degree of freedom given by uniform rotations, it is customary to fix $\theta_0 = 0$. Then such synchronization configurations are characterized by the system of n

nonlinear equations,

$$\omega_i - \sum_{j \in \mathcal{N}_G(i)} a_{ij} \sin(\theta_i - \theta_j) = 0 \quad \text{for } i = 1, \dots, n, \tag{2}$$

in the variables $\theta_1, \dots, \theta_n$ with constant $\theta_0 = 0$. Then, the root counting problem is the following.

Problem Statement 1 (real root count). *Given $\omega_1, \dots, \omega_n \in \mathbb{R}$ and a weighted graph of $n+1$ nodes, what is the maximum number of real roots the induced system (2) could have?*

An upper bound to this answer, that is independent from network topology, is shown to be $\binom{2n}{n}$ [2]. However, recent studies [13, 36] suggest that much tighter upper bounds that are sensitive to network topology may exist. In this paper, we show that this is true.

To leverage tools from algebraic geometry, we reformulate the synchronization system (2) as a system of rational equations. Using the identity $\sin(\theta_i - \theta_j) = \frac{1}{2\mathbf{i}}(e^{\mathbf{i}(\theta_i - \theta_j)} - e^{-\mathbf{i}(\theta_i - \theta_j)})$ where $\mathbf{i} = \sqrt{-1}$, (2) can be transformed into

$$\omega_i - \sum_{j \in \mathcal{N}_G(i)} \frac{a_{i,j}}{2\mathbf{i}} (e^{\mathbf{i}\theta_i} e^{-\mathbf{i}\theta_j} - e^{-\mathbf{i}\theta_i} e^{\mathbf{i}\theta_j}) = 0 \quad \text{for } i = 1, \dots, n.$$

With the substitution $x_i := e^{\mathbf{i}\theta_i}$ for $i = 1, \dots, n$, we obtain the Laurent polynomial system

$$F_{G,i}(x_1, \dots, x_n) = \omega_i - \sum_{j \in \mathcal{N}_G(i)} a'_{ij} \left(\frac{x_i}{x_j} - \frac{x_j}{x_i} \right) = 0 \quad \text{for } i = 1, \dots, n, \tag{3}$$

where $a'_{ij} = \frac{a_{ij}}{2\mathbf{i}}$ and $x_0 = 1$ is a constant. The system $F_G = (F_{G,1}, \dots, F_{G,n})$, is a system of n rational equations in the n complex variables $\mathbf{x} = (x_1, \dots, x_n)$. Since x_i's appear in the denominator positions, F_G is defined only on $(\mathbb{C}^*)^n = (\mathbb{C} \setminus \{0\})^n$. Clearly, each equivalence class of real roots of (2) (modulo translations by multiples of 2π) corresponds to a root of (3) in $(\mathbb{C}^*)^n$. Therefore, we can consider a more general root counting problem as follows.

Problem Statement 2 ($\mathbb{C}^*$-root count problem). *Given complex constants $\omega_1, \dots, \omega_n$ and a weighted graph of $n+1$ nodes with weights $\{a'_{ij}\}$, what is the maximum number of isolated $\mathbb{C}^*$-roots the system (3) could have?*

In this context, Problem 1 will be a specialized version of Problem 2 since the real roots counted in the first problem correspond to roots in the real torus $(S^1)^n \subset (\mathbb{C}^*)^n$, where $S^1 = \{e^{\mathbf{i}\theta} \mid \theta \in \mathbb{R}\}$ is the unit circle on the complex plane. Consequently, the root count we obtain for Problem 2 will also be an upper bound for the root count in Problem 1. The benefit of focusing on Problem 2 is that the algebraic formulation for Problem 2 allows the use of powerful tools from complex algebraic geometry, in particular, the theory of the birationally invariant intersection index.

An important consequence of studying the root count over $\mathbb{C}^*$ is that achieving the maximum root count is also a "generic behavior" in the entire family of Kuramoto systems induced by the same network topology. In general, given n complex vector spaces $L_1, \dots, L_n$ of rational functions in n variables, Kaveh and Khovanskii [29] established that for generic choices

$f_1 \in L_1, \ldots, f_n \in L_n$, the number of common isolated complex roots of $f_1, \ldots, f_n$ in $(\mathbb{C}^*)^n$ (or more general toric varieties) is a fixed number, known as the *birationally invariant intersection index* of $L_1, \ldots, L_n$ in $(\mathbb{C}^*)^n$ which will be denoted by $[L_1, \ldots, L_n]$ [29, Definition 4.5 and Theorem 4.6]. This number also coincides with the maximum common isolated complex roots that any choices of $f_1 \in L_1, \ldots, f_n \in L_n$ could have [22, 40]. Moreover, $[L_1, \ldots, L_n]$ is given by the mixed volume of Newton–Okounkov bodies associated with $L_1, \ldots, L_n$ and hence is a wide generalization of the well-known Bernstein–Khovanskii–Kushnirenko (BKK) bound [5, 30, 33]. We show that the root count in Problem 2 is a special case of the birationally invariant intersection index. For each vertex $i = 1, \ldots, n$, define the complex vector space of rational functions

$$L_{G,i} = \operatorname{span}\left(\{1\} \cup \{x_i x_j^{-1} - x_i^{-1} x_j\}_{j \in \mathcal{N}_G(i)}\right). \tag{4}$$

With this construction, the ith function in (3) is an element in $L_{G,i}$. Therefore, the number of $\mathbb{C}^*$-roots of (3) for generic choices of weights and constant terms will be equal to the number of common roots of n generic elements from $L_{G,1}, \ldots, L_{G,n}$, respectively, within $(\mathbb{C}^*)^n$. This is precisely the birationally invariant intersection index [29], denoted $[L_{G,1}, \ldots, L_{G,n}]$.

Problem Statement 3 (birationally invariant intersection index). *Given a graph G with $n+1$ vertices $0, 1, \ldots, n$, let $L_{G,i} = \operatorname{span}(\{1\} \cup \{x_i x_j^{-1} - x_i^{-1} x_j\}_{j \in \mathcal{N}(i)})$. What is $[L_{G,1}, \ldots, L_{G,n}]$?*

Note that if the coupling strengths $\{a_{ij}\}$ are considered to be independent, then Problems 2 and 3 will be equivalent. The usual convention, however, is to assume that $\{a_{ij}\}$ is symmetric, i.e., $a_{ij} = a_{ji}$. In that case, it is not immediately obvious that these two problems are still equivalent. As we shall demonstrate (Lemma 8), even under the assumption $a_{ij} = a_{ji}$, these two questions still yield the exact same answer for tree graphs.

Though the intersection index $[L_{G,1}, \ldots, L_{G,n}]$ can be expressed as the generalized mixed volume of the Newton–Okounkov bodies associated with $L_{G,1}, \ldots, L_{G,n}$, its direct computation, in general, remains a difficult problem. Using a construction known as the "adjacency polytope bound" developed in [11, 14], we will show for trees and cycles that $[L_{G,1}, \ldots, L_{G,n}]$ will be 2^{N-1} and $N\binom{N-1}{\lfloor (N-1)/2 \rfloor}$, respectively. Both are significantly less than the only known upper bound $\binom{2N-2}{N-1}$ for the general case (heterogeneous oscillators with nonuniform coupling) of the Kuramoto equations (2) even for small values of n. Asymptotically, in both cases the ratio between the new bounds and $\binom{2N-2}{N-1}$ goes to zero as $N \to \infty$. Moreover, the intersection index derived from tree graphs also coincides with the well-known lower bound of the number of *real* roots to the original (nonalgebraic) system (2), showing that the intersection index derived from a complex root count can actually be a sharp bound for real roots.

3. Kuramoto model and synchronization equations. The study of synchronization in networks of coupled oscillators is a particularly pervasive subject in a wide range of independent fields of study in biology, physics, chemistry, engineering, and social science. We refer the reader to [1, 6, 47] for a detailed historical account of this topic. The simplest mechanical analogue of the coupled oscillator model (2) is a spring network, shown in Figure 1, that consists of a set of weightless particles constrained to move on the unit circle without friction or

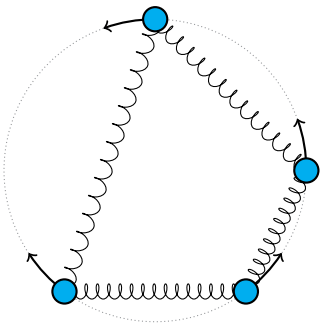

Figure 1. *A spring network.*

collision [19]. Here, the coupling strength[1] $a_{ij} > 0$ characterizes the stiffness of the spring connecting particles i and j, and $\frac{d\theta_i}{dt}$ represents the angular velocity (or equivalently, frequency) of the ith particle. Of great interest is the configuration in which the angular velocity of *all* particles can become perfectly aligned, known as *frequency synchronization*. That is, $\frac{d\theta_i}{dt} = c$ for $i = 0, \ldots, n$ and a constant c. Adopting a rotational frame of reference, we can always assume $c = 0$. That is, frequency synchronization configurations are equivalent to equilibria of the Kuramoto model (1). Under this assumption, the $n+1$ equilibrium equations must sum to zero. This allows for the elimination of one of the equations, producing the system (2) of n equations in n unknowns. Despite its mechanical origin, the frequency synchronization system (2) naturally appears in a long list of seemingly unrelated fields, including electrical power networks [2, 20], flocking behavior in biology and control theory [27, 48], and decentralized clock synchronization [45]. We refer the reader to [19] for a detailed list.

In [2], an upper bound on the number of equilibria of the Kuramoto model (solutions to (2)) induced by a graph of N vertices with any coupling strengths is shown to be $\binom{2N-2}{N-1}$. For certain cases, such as the Kuramoto model on the one-, two-, and three-dimensional lattice graphs with different boundary conditions, as well as for complete and planar graphs, all or at least a class of equilibria were analytically [10, 17, 18, 28, 38, 41, 43, 50] and numerically [25, 26, 34, 37, 49] found in previous studies. For tree graphs of N nodes, it is well known that there could be as many as 2^{N-1} real equilibria. Various algebraic formulations have been used to leverage results from algebraic geometry and numerically find some or all equilibria for certain small graphs [13, 35, 36, 39]. Recently, in the special case of "rank-one coupling," i.e., the matrix $[a_{ij}]$ has rank 1, a much smaller bound $2^N - 2$ was established [15]. Based on the theory of the BKK bound, a search for topology-dependent bounds on the number of solutions to (2) and (3) was initiated in [13, 36]. In the present contribution, we provide explicit formulas for a much stronger solution bound: the birationally invariant intersection index.

4. Preliminaries and notation. For a compact set $Q \subset \mathbb{R}^n$, $\mathrm{vol}_n(Q)$ denotes its standard Euclidean volume, and the quantity $n!\,\mathrm{vol}_n(Q)$ is its *normalized volume*, denoted $\mathrm{NVol}_n(Q)$. Say Q is *convex* if it contains the line segment connecting any two points Q. For a set $X \subset \mathbb{R}^n$,

[1] In the original model proposed by Kuramoto, the coupling strengths are symmetric, i.e., $a_{ij} = a_{ji}$. However, in more general applications (such as power-flow equations), perfect symmetry may not hold.

its *convex hull* is the smallest convex set containing it, denoted $\mathrm{conv}(X)$, and its *affine span* is the smallest affine subspace of $\mathbb{R}^n$ containing it, denoted $\mathrm{aff}(X)$. A (*convex*) *polytope* is the convex hull of a finite set of points. Of particular importance in the current context are convex polytopes whose vertices lie in $\mathbb{Z}^n$. Such polytopes are called *lattice polytopes*. A full-dimensional convex lattice polytope $P \subset \mathbb{R}^n$ is said to be *reflexive* if its dual

$$P^* = \{\mathbf{x} \in \mathbb{R}^n \mid \langle \mathbf{x}, \mathbf{p} \rangle \geq -1 \ \forall \mathbf{p} \in P\}$$

is also a lattice polytope. Given two convex polytopes $P \subset \mathbb{R}^n$ and $Q \subset \mathbb{R}^m$ both containing the origin, their *free sum*, denoted $P \oplus Q$, is $\mathrm{conv}(P' \cup Q') \subset \mathbb{R}^{n+m}$, where

$$P' = \{(\mathbf{p}, \mathbf{0}) \in \mathbb{R}^{n+m} \mid \mathbf{p} \in P\}$$

and

$$Q' = \{(\mathbf{0}, \mathbf{q}) \in \mathbb{R}^{n+m} \mid \mathbf{q} \in Q\}.$$

An important fact is that under mild conditions, the normalized volume of a free sum of lattice polytopes factors as the product of normalized volumes of the summand polytopes.

Lemma 1 ([7, Theorem 1]). *Given two convex lattice polytopes P and Q both containing the origin as an interior point, if one is reflexive, then* $\mathrm{NVol}(P \oplus Q) = \mathrm{NVol}(P) \cdot \mathrm{NVol}(Q)$.

The set $(\mathbb{C}^*)^n$, known as an *algebraic torus*, has the structure of an Abelian group under componentwise multiplication, and it will be the space in which we study the root count of synchronization equations. A *Laurent monomial* in $\mathbf{x} = (x_1, \ldots, x_n)$ induced by vector $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{Z}^n$ is the formal expression $\mathbf{x}^{\mathbf{a}} = x_1^{a_1} \cdots x_n^{a_n}$. It is easy to verify that as a map from $(\mathbb{C}^*)^n$ to $\mathbb{C}^*$, $\mathbf{x}^{\mathbf{a}}$ is actually a character, i.e., a group homomorphism. In general, a system of Laurent monomials induced by $\mathbf{a}_1, \ldots, \mathbf{a}_m \in \mathbb{Z}^n$ gives rise to the group homomorphism $\mathbf{x} \mapsto (\mathbf{x}^{\mathbf{a}_1}, \ldots, \mathbf{x}^{\mathbf{a}_m})$ between $(\mathbb{C}^*)^n$ and $(\mathbb{C}^*)^m$. Of particular importance is the case where $m = n$.

Lemma 2 ([22]). *Given vectors $\mathbf{a}_1, \ldots, \mathbf{a}_n \in \mathbb{Z}^n$, the map $\mathbf{x} \mapsto (\mathbf{x}^{\mathbf{a}_1}, \ldots, \mathbf{x}^{\mathbf{a}_n})$ is an automorphism of $(\mathbb{C}^*)^n$ if and only if $|\det[\mathbf{a}_1, \ldots, \mathbf{a}_n]| = 1$, and in that case the map is a biholomorphism.*

For the integer matrix $A = [\mathbf{a}_1, \ldots, \mathbf{a}_n]$ and $\mathbf{x} = (x_1, \ldots, x_n)$ above, we use the compact notation $\mathbf{x}^A = (\mathbf{x}^{\mathbf{a}_1}, \ldots, \mathbf{x}^{\mathbf{a}_n})$ to represent the automorphism induced by A. Such a square integer matrix A with $|\det(A)| = 1$ is said to be *unimodular*. Integer matrices are of particular importance because the linear transformations they induce, known as *unimodular transformations*, preserve normalized volume. More generally, an integer matrix (not necessarily square) is *totally unimodular* if all its nonsingular submatrices are unimodular. This concept also extends to lattice polytopes: A lattice simplex is *unimodular* if its normalized volume is 1, and a simplicial subdivision of a lattice polytope is *unimodular* if it consists of only unimodular simplices. Two polytopes are *unimodularly* equivalent if one is the image of the other under a unimodular transformation, and they necessarily have the same normalized volume.

A *Laurent polynomial* is a finite linear combination of distinct Laurent monomials, i.e., an expression of the form $f = \sum_{\mathbf{a} \in S} c_{\mathbf{a}} \mathbf{x}^{\mathbf{a}}$ for some finite $S \subset \mathbb{Z}^n$. The set $\mathrm{conv}(S) \subset \mathbb{R}^n$ is called the *Newton polytope* of f. Newton polytopes play a critical role in calculating the

generic number of isolated roots in $(\mathbb{C}^*)^n$ (or simply $\mathbb{C}^*$-roots) that a system of n Laurent polynomials could have. Indeed, this generic $\mathbb{C}^*$-root count is given by the mixed volume of the Newton polytopes. This is the content of Bernshtein's theorem [5, 30, 33], and this count has since been known as the *BKK bound* [9].

Remark 1. "Genericity" is a fundamental concept in algebraic geometry: A generic property is true for almost all members of a certain family. In our current context, we can take the following "probability 1" interpretation for the above lemma: If the coefficients of $(f_1, \dots, f_n)$ are chosen at random (with independent distributions) among all possible complex coefficients, then with probability 1, the BKK bound is exact. Stated more precisely, within the space of all possible coefficients, there is a nonempty Zariski open set such that the BKK bound is exact for all choices of coefficients in this set. Here, a Zariski open set in this space is a complement of the zero set of a nontrivial polynomial in the coefficients. Such a Zariski open set is necessarily open and everywhere dense. The same interpretation applies to the definition of the birationally invariant intersection index.

A relaxation of the BKK bound was developed in the context of algebraic synchronization equations [11] as well as the closely related "power-flow equations" [14].

Definition 3 (adjacency polytope). *Given a graph G, we define its* adjacency polytope *as*

$$\nabla_G = \operatorname{conv}(\{\pm(\mathbf{e}_i - \mathbf{e}_j) \mid (i,j) \in E(G)\}).$$

The normalized volume $\operatorname{NVol}(\nabla_G)$ *is called the* adjacency polytope bound *of G.*

The polytope ∇_G is essentially the convex hull of the union of the Newton polytopes of the n Laurent polynomial equations in (3), and it can be considered as a geometric encoding of the topology of the graph G. Adjacency polytopes have been previously studied in order to identify properties of a related semigroup algebra, such as in [44]. However, previous work has not addressed the normalized volume of these polytopes. A simple observation [11, 14] is that the adjacency polytope bound (or simply, *AP bound*) is indeed an upper bound for answers to Problems 2 and 3.

Proposition 4. *Given a graph G containing vertices $\{0, 1, \dots, n\}$, the number of isolated $\mathbb{C}^*$-solutions for the algebraic system (3) is bounded by the AP bound* $\operatorname{NVol}(\nabla_G)$.

By comparing the constructions of the root counts outlined above, it is easy to verify the following chain of inequalities:

$$\begin{matrix}\mathbb{R}\text{-root} \\ \text{count of (2)}\end{matrix} \leq \begin{matrix}\mathbb{C}^*\text{-root} \\ \text{count of (3)}\end{matrix} \leq [L_{G,1}, \dots, L_{G,n}] \leq \begin{matrix}\text{BKK} \\ \text{bound}\end{matrix} \leq \begin{matrix}\text{AP} \\ \text{bound}\end{matrix}. \tag{5}$$

5. Tree graphs. This section provides the answers to Problems 2 and 3 for a tree graph T_N containing $N = n + 1$ vertices. The strategy is to bound $[L_{T_N,1}, \dots, L_{T_N,n}]$ from above using the AP bound, and then bound it from below by examining the actual number of roots. With this, we show that$[L_{T_N,1}, \dots, L_{T_N,n}]$ is $2^n = 2^{N-1}$. This agrees with a well-known fact in the study of the Kuramoto model: For tree graphs, the original (nonalgebraic) Kuramoto model (1) could have as many as 2^{N-1} real equilibria. This shows that even though it is derived from a complex algebraic formulation, the bound $[L_{T_N,1}, \dots, L_{T_N,n}]$ is also a sharp

bound for the real roots. That is, *the algebraic reformulation* (3) *from* (2) *and the extension to the field of complex numbers does not alter the maximum root count.*

Following the common convention, we fix vertex 0 to be the root vertex of T_N. For any other vertex i, let $\pi(i)$ be the unique parent vertex of i, and let $d(i)$ be the depth of the vertex i (the number of edges in the path between vertex i and vertex 0). We introduce a new set of variables $y_1, \ldots, y_n$ associated with the nonroot vertices and consider the change of variables that replaces each x_i by the product of all y_j's associated to vertices on the path between i and the root. This construction can be considered as an algebraic analogue of the per-path analysis that is often used in the study of the Kuramoto model [19]. We show that this change of variables preserves all the properties of the solution set for (3) that are relevant in our discussion.

Lemma 5. *The map* $\phi = (\phi_1, \ldots, \phi_n) : (\mathbb{C}^*)^n \to (\mathbb{C}^*)^n$ *given by*

$$\phi_i(y_1, \ldots, y_n) = y_i \prod_{k=1}^{d(i)-1} y_{\pi^k(i)} \quad \text{for } i = 1, \ldots, n$$

is a bijection, and the Jacobian matrix $D\phi$ *is nonsingular everywhere.*

Proof. A tree, by definition, has no cycles, so it is always possible to re-index the vertices such that vertex 0 is the root and $\pi(i) < i$ for any i. With this convention, we can write ϕ as $\phi(\mathbf{y}) = y^A$ where $y = (y_1, \ldots, y_n)$, and A is an $n \times n$ upper triangular integer matrix with all diagonal entries being 1. Then A is a unimodular matrix, and hence A^{-1} is also a unimodular integer matrix. It is easy to verify that $\psi(\mathbf{x}) = \mathbf{x}^{A^{-1}}$ is an inverse of ϕ, and therefore both ψ and ϕ bijections. Moreover, since $\det A = 1$, by Lemma 2 $D\phi(\mathbf{y})$ is nonsingular for all $\mathbf{y} \in (\mathbb{C}^*)^n$. ∎

Being a bijection, the transformation ϕ given in Lemma 5 preserves the $\mathbb{C}^*$-root count of any system of Laurent polynomials. Moreover, since $D\phi$ remains nonsingular on $(\mathbb{C}^*)^n$, ϕ also preserves the more subtle local structures at each solution including multiplicities and local dimensions. Therefore, in the following we answer the root counting question in the $\mathbf{y}$-coordinate system via the nonlinear change of variables $\mathbf{x} = \phi(\mathbf{y})$.

Theorem 6. *For a tree graph* T_N *consisting of* N *nodes, the AP bound of the induced algebraic system* (3) *is* 2^{N-1}.

This result agrees with the general analysis from recent studies [16, 19]. A similar result for the root counting problem for power-flow equations was developed in [23].

Proof. Let $F_{T_N}(\mathbf{x}) = F_{T_N}(x_1, \ldots, x_n)$ be the algebraic system (3) induced by the tree graph T_N. Then each nonconstant monomial in $F_{T_N}(\mathbf{x})$ must be of the form $x_i x_{\pi(i)}^{-1}$ or $x_i^{-1} x_{\pi(i)}$ for some $i \in \{1, \ldots, n\}$. With the substitution $\mathbf{x} = \phi(\mathbf{y})$ as defined in the previous lemma, it is easy to verify that the nonlinear interactions among the variables in F_{T_N} are "decoupled" in the sense that

$$x_i x_{\pi(i)}^{-1} = \left(y_i \prod_{k=1}^{d(i)-1} y_{\pi^k(i)} \right) \left(y_{\pi(i)}^{-1} \prod_{k=1}^{d(\pi(i))-1} y_{\pi^k(\pi(i))}^{-1} \right) = y_i.$$

Therefore the set of monomials which appear in $F_{T_N}(\phi(\mathbf{y}))$ is exactly the set $\{1\}\cup\{y_1,\dots,y_n\}\cup\{y_1^{-1},\dots,y_n^{-1}\}$. Under the same transformation, the AP becomes the unimodularly equivalent cross-polytope

$$\operatorname{conv}\left(\bigcup_{i=1}^{n}\operatorname{conv}(\{\pm\mathbf{e}_i\})\right),$$

which is a free sum of the n line segments $\operatorname{conv}(\{\pm\mathbf{e}_i\})$ for $i=1,\dots,n$. By Lemma 1, the normalized volume of this polytope is the product of the normalized volume of each of the summands. Since each line segment is of length 2, the AP bound is therefore $2^n = 2^{N-1}$. ∎

We now show that the AP bound is actually sharp. That is, there exist choices of complex values for $\{a'_{ij}\}$ and $\omega_1,\dots,\omega_n$ in (3) for which the system has exactly 2^n isolated $\mathbb{C}^*$-roots.

Lemma 7. *For the tree graph T_N containing $N=n+1$ vertices, the induced algebraic system $F_{T_N}(\phi(\mathbf{y}))$ is equivalent to the system*

$$\omega_i^* - a'_{i,\pi(i)}\left(y_i - y_i^{-1}\right) = 0 \quad \text{for } i=1,\dots,n \tag{6}$$

for some complex constants $\omega_1^*,\dots,\omega_n^*$.

Here, the equivalence means the two systems have the same solution set in $\mathbb{C}^*$.

Proof. For $N=2$, the system $F_{T_N}(\phi(\mathbf{y}))$ contains only one equation,

$$\omega_i - a'_{1,0}(y_1 - y_1^{-1}) = 0,$$

where $y_0 = 1$. The statement is obviously true in this case.

Now consider a tree T_N consisting of $N=n+1$ nodes, and assume the statement is true for any tree of smaller size. Fixing any leaf vertex in the tree, without loss of generality we can re-index the vertices so that this leaf vertex has index n and its unique parent vertex is $n-1$. In this arrangement, the nth (last) equation in (3) is

$$\omega_n - a'_{n,n-1}(y_n - y_n^{-1}) = 0, \tag{7}$$

while the $(n-1)$th equation is

$$\omega_{n-1} - a'_{n-1,n}(y_n^{-1} - y_n) - a_{n-1,\pi(n-1)}(y_{n-1} - y_{n-1}^{-1}) - \sum_{j\in D} a'_{n-1,j}(y_j^{-1} - y_j) = 0, \tag{8}$$

where $D = \mathcal{N}(n-1)\setminus\{n,\pi(n-1)\}$. Then adding $a'_{n-1,n}/a'_{n,n-1}$ times (7) to (8) produces

$$\left(\omega_{n-1} + \frac{a'_{n-1,n}}{a'_{n,n-1}}\omega_n\right) - a_{n-1,\pi(n-1)}(y_{n-1} - y_{n-1}^{-1}) - \sum_{j\in D} a'_{n-1,j}(y_j^{-1} - y_j) = 0.$$

With this transformation, the first $n-1$ equations do not involve y_n, and they form a smaller algebraic system induced by a tree graph consisting of n vertices $0,1,\dots,n-1$. By the induction hypothesis, this smaller system can be transformed into the desired form given in (6) without altering the solution set. By induction, the statement is true for all tree graphs. ∎

Lemma 8. *Given a tree graph T_N containing N vertices, there exist choices of complex valued weights $\{a'_{ij}\}_{(i,j)\in E(T_N)}$ and complex constants $\omega_1,\dots,\omega_n$, such that the induced system $F_{T_N}(\mathbf{x}) = \mathbf{0}$ has exactly 2^{N-1} nonsingular isolated $\mathbb{C}^*$-solutions.*

Proof. By Lemma 7, the induced algebraic system $F_{T_N}(\mathbf{x})$ is equivalent to (6). Under the transformation $\mathbf{x} = \phi(\mathbf{y})$ given in Lemma 5, $F_{T_N}(\phi(\mathbf{y}))$ is

$$\omega_i^* - a'_{i,\pi(i)}\,(y_i - y_i^{-1}) = 0 \qquad \text{for } i = 1,\dots,n, \tag{9}$$

which has the same number of isolated nonsingular solutions in $(\mathbb{C}^*)^n$ as the original system. Concerning $\mathbb{C}^*$-solutions, the ith equation in the above system is equivalent to the quadratic equation

$$\omega_i^* y_i - a'_{i,\pi(i)}\,y_i^2 + a'_{i,\pi(i)} = 0,$$

which has exactly two $\mathbb{C}^*$-solutions for a generic choice of coefficients (even if we require $a'_{ij} = a'_{ji}$). Since there are n independent quadratic equations in $y_1,\dots,y_n$, respectively, the generic root count for (9) is exactly $2^n = 2^{N-1}$. Consequently the $\mathbb{C}^*$-solution count of the original system F_{T_N} can also reach 2^{N-1}. ∎

Recall that the intersection index $[L_{T_N,1},\dots,L_{T_N,n}]$ is an upper bound for the number of isolated $\mathbb{C}^*$-solutions that the system F_{T_N} could have. Therefore $[L_{T_N,1},\dots,L_{T_N,n}]$ is at least 2^{N-1}. Although this lemma does not directly demonstrate that generic choices of coefficients for (9) correspond to a generic choice of F_{T_N}, by combining Theorem 6 and the comparison (5), we can conclude that $[L_{T_N,1},\dots,L_{T_N,n}] = 2^{N-1}$.

Corollary 9. *Given a tree graph T_N containing $N = n+1$ vertices, let $L_{T_N,1},\dots,L_{T_N,n}$ be the subspace of rational functions defined in (4). Then*

$$[\,L_{T_N,1},\dots,L_{T_N,n}\,] = 2^n = 2^{N-1}.$$

6. Cycle graphs. In the study of the Kuramoto model, cycle graphs may be considered as basic building blocks; recent works suggest that it is plausible that detailed analysis of the local geometry near equilibria can be done on a cycle-by-cycle basis [8]. In the context of power-flow study, the analysis of the Kuramoto model on cycle graphs is also of great practical importance [49].

For a cycle graph C_N of $N = n+1$ vertices (labeled by $\{0,\dots,n\}$), we show that the intersection index $[L_{C_N,1},\dots,L_{C_N,n}]$ is $(n+1)\binom{n}{\lfloor n/2\rfloor}$. Following the strategy used in the previous section, we first compute the AP bound for the cycle graph C_N. Then we show that there is no gap between $[L_{C_N,1},\dots,L_{C_N,n}]$ and the AP bound.

The set of edges is $E(C_N) = \{(0,1),(1,2),\dots,(n-1,n),(n,0)\}$. The induced adjacency polytope (Definition 3) is

$$\nabla_{C_N} = \operatorname{conv}\{\pm(\mathbf{e}_i - \mathbf{e}_j) \mid (i,j) \in E(C_N)\},$$

where $\mathbf{e}_0 = (0,\dots,0)$ as before. The AP bound for F_{C_N} is defined as the normalized volume of ∇_{C_N}; thus, the first goal of this section is to identify this normalized volume. It will be simplest to first notice that ∇_{C_N} is unimodularly equivalent to the polytope

$$P_N = \operatorname{conv}\{\pm\mathbf{e}_1\,,\ \dots\ ,\ \pm\mathbf{e}_n\,,\ \pm(\mathbf{e}_1 + \cdots + \mathbf{e}_n)\}.$$

Such an equivalence can be seen by applying the normalized volume-preserving (unimodular) transformation given by

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ 1 & 1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 1 \end{bmatrix}$$

to each vertex of ∇_{C_N}. One reason this is desirable is because it becomes clear that P_N is totally unimodular; that is, the matrix formed by placing the vertices of P_N as the columns is a totally unimodular matrix. Since $\mathbf{0}$ is the average of all vertices of P_N, it is an interior point of P_N. Thus, a unimodular triangulation of the boundary of P_N will induce a unimodular triangulation of P_N itself, where the simplices are of the form $\operatorname{conv}\{\mathbf{0} \cup \Delta\}$, where Δ is a simplex in the triangulation of the boundary. This will be our strategy, since the number of simplices in a unimodular triangulation of a polytope is identical to the normalized volume of the polytope.

When N is odd, P_N is called a *del Pezzo* polytope. In this case, it is known [42, Remark 4.3] that P_N is simplicial; that is, every facet is a simplex. Together with P_N being totally unimodular, its normalized volume is therefore equal to the number of its facets, which was shown to be $N\binom{N-1}{(N-1)/2}$. So, we need only consider when N is even, and we would like to obtain an analogous formula.

Proposition 10. *For even N, let $\Lambda_n \subseteq \{-1,1\}^n$ be the set of sequences $(\lambda_1,\dots,\lambda_n)$ such that $\sum_{i=1}^n \lambda_i = 1$. The facets of P_N are then*

$$\mathcal{F}(P_N) = \left\{ \pm \operatorname{conv}\left\{ \lambda_1 \mathbf{e}_1, \dots, \lambda_n \mathbf{e}_n, \sum_{i=1}^n \mathbf{e}_i \right\} \mid (\lambda_1,\dots,\lambda_n) \in \Lambda_n \right\}.$$

Proof. First, observe that the vertices of a facet must consist of a subset of

$$\{\lambda_1 \mathbf{e}_1, \dots, \lambda_n \mathbf{e}_n, \lambda_{n+1}(\mathbf{e}_1 + \cdots + \mathbf{e}_n)\} \tag{10}$$

for some choice of $\lambda_1,\dots,\lambda_{n+1} \in \{-1,1\}$. Otherwise, two vertices $\pm\mathbf{v}$ of P_N would be part of a facet, which is impossible since the line segment $\operatorname{conv}\{-\mathbf{v},\mathbf{v}\}$ passes through the interior of P_N.

Next, note that if F is a facet, then so is $-F$ since $P_N = -P_N$. One specific choice of facet is

$$F_0 = \operatorname{conv}\left\{ \mathbf{e}_1, \dots, \mathbf{e}_{(n-1)/2}, -\mathbf{e}_{(n+1)/2}, \dots, -\mathbf{e}_n, -\sum_{i=1}^n \mathbf{e}_i \right\}.$$

To see why this is true, observe that each of the vertices in F_0 lies on the hyperplane

$$\{(x_1 \dots, x_n) \in \mathbb{R}^n \mid \ell(x_1,\dots,x_n) = 1\},$$

where

$$\ell(x_1,\dots,x_n) = \sum_{i=1}^{(n-1)/2} x_i - \sum_{i=(n+1)/2}^{n} x_i,$$

and all other vertices $\mathbf{v}$ of P_N satisfy $\ell(\mathbf{v}) = -1$. Moreover, the first n vertices defining F_0 are clearly affinely independent, so $\dim F_0 = n-1$. Therefore, F_0 is indeed a facet of P_N.

Any other choice of $(\lambda_1, \ldots, \lambda_n) \in \Lambda_n$ for the elements in $\mathcal{F}(P_N)$ will result in a facet as well, since the resulting convex hull is unimodularly equivalent to F_0. Hence, the same arguments can be applied to these sets. It remains to show that no other set of vertices will form a facet.

Take any element of (10) such that there are $k \geq 2$ more negative than positive coefficients on the summands $\mathbf{e}_1, \ldots, \mathbf{e}_n$, and set $\lambda_{n+1} = -1$. Without loss of generality, we can assume $\lambda_1 = \cdots = \lambda_{(n-2k+1)/2} = 1$ and the remaining $\lambda_i = -1$. Call their convex hull F'. Form $\ell(x)$ as before and note that the first n vertices of F' satisfy $\ell(x_1, \ldots, x_n) = 1$. Additionally, the vertices $\mathbf{v}$ of P_N not in F' satisfy $\ell(x_1, \ldots, x_n) < 1$. However, $\ell(-\mathbf{e}_1 - \cdots - \mathbf{e}_n) = k$, so $\operatorname{aff}(F')$ actually passes through the interior of P_N and cannot define a facet.

Note as well that if we take any n-element subset of (10) without $\pm(\mathbf{e}_1 + \cdots + \mathbf{e}_n)$, then we come across a problem similar to that in the previous paragraph. If we take an n-element subset that excludes $\pm\mathbf{e}_j$ for some j, then the resulting hyperplane is exactly the same as if we included $\pm\mathbf{e}_j$. Therefore, there are no facets of any other form. ∎

In order to continue, we need to define more terminology. Given an element $p \in \mathbb{Q}^n$, let $N_p = \mathbb{Z}(\mathbf{e}_1, \ldots, \mathbf{e}_n, p)$, a lattice in $\mathbb{R}^n$, where $\mathbb{Z}A$ indicates the set of $\mathbb{Z}$-linear combinations of elements of A. We say that two sublattices $J, K \subseteq N_p$ are *complementary* if every element of $\operatorname{aff}(J \cup K) \cap N_p$ is the sum of a unique element of J and a unique element of K. Additionally, a polytope $P \subseteq \mathbb{R}^n$ is called an *affine free sum* if $P = \operatorname{conv}(P_1 \cup P_2)$, where $P_1 \cap P_2 = \{p\}$ for some $p \in \mathbb{Q}^n$, and the two lattices $(\operatorname{aff}(P_1 - p)) \cap N_p$ and $(\operatorname{aff}(P_2 - p)) \cap N_p$ are complementary. Finally, we say that $P \subseteq \mathbb{R}^n$ is *Gorenstein of index* k if

1. there exists a smallest positive integer k for which kP contains a unique lattice point in its interior;
2. P is a lattice polytope; and
3. the *polar dual* of P,
$$\{x \in \mathbb{R}^k \mid x^T y \leq 1 \text{ for all } y \in P\},$$
is also a lattice polytope.

If P is a translate of a Gorenstein polytope by a lattice point, then we also call P Gorenstein.

By setting $\mathbf{z} = (1, \ldots, 1, t)$ as in [3, Corollary 5.9] and applying [4, Corollary 3.21], we can see all of these definitions come together nicely in the following way.

Lemma 11. *If $P \subseteq \mathbb{R}^n$ is the affine free sum of P_1 and P_2, and P_1 is Gorenstein of index 1, then*
$$\operatorname{NVol}(P) = \operatorname{NVol}(P_1) \operatorname{NVol}(P_2).$$

A more direct proof of this result is also presented in the recent work [12] by the first two authors.

We can now take another step toward finding the normalized volume of P_N.

Proposition 12. *The normalized volume of each facet of $P(C_N)$ is $\frac{N}{2}$.*

Proof. By permuting the coordinates of each facet of P_N, which are described in Proposition 10, one sees that the elements of $\mathcal{F}(P_N)$ are all unimodularly equivalent to one another.

Since unimodular equivalence preserves normalized volume, it is sufficient to find the normalized volume of only one facet; for convenience, we will select the facet F_0 from the previous proof.

Our overall strategy will be to first project F_0 into a smaller-dimensional space in a way that preserves the normalized volume. Then, we will construct a triangulation of this projection into unimodular simplices, and finally, count the number of simplices in the triangulation.

By applying the unimodular transformation $x \mapsto Ax$ to F_0, where $A = (a_{i,j})$ is the $n \times n$ matrix

$$a_{i,j} = \begin{cases} 1 & \text{if } i = j \text{ or both } i = n,\ j < (n-1)/2, \\ -1 & \text{if both } i = n,\ (n-1)/2 < j < n, \\ 0 & \text{else,} \end{cases}$$

we obtain a polytope whose vertices are identical to those of F_0 in the first $n-1$ coordinates and are exactly 1 in the final coordinate. Since this is a unimodular map, $\mathrm{NVol}(F_0) = \mathrm{NVol}(f(F_0))$. By projecting $f(F_0)$ to the first $n-1$ coordinates, we obtain the polytope $\overline{F}_0$, which also satisfies $\mathrm{NVol}(F_0) = \mathrm{NVol}(\overline{F}_0)$. As a result, we have

$$\overline{F}_0 = \mathrm{conv}\left\{\mathbf{0}, \mathbf{e}_1, \ldots, \mathbf{e}_{(n-1)/2}, -\mathbf{e}_{(n+1)/2}, \ldots, -\mathbf{e}_{n-1}, -\sum_{i=1}^{n-1} \mathbf{e}_i\right\} \subseteq \mathbb{R}^{n-1}.$$

Notice that we can write $\overline{F}_0 = \mathrm{conv}\{G_1 \cup G_2\}$, where

$$G_1 = \mathrm{conv}\left\{\mathbf{e}_1, \ldots, \mathbf{e}_{(n-1)/2}, -\sum_{i=1}^{n-1} \mathbf{e}_i\right\}$$

and

$$G_2 = \mathrm{conv}\{\mathbf{0}, -\mathbf{e}_{(n+1)/2}, \ldots, -\mathbf{e}_{n-1}\},$$

which intersect only at the origin. Additionally, the intersection of their affine spans is the single point

$$\{\mathbf{v}_0\} = \mathrm{aff}(G_1) \cap \mathrm{aff}(G_2) = \left\{\left(0, \ldots, 0, -\frac{1}{(n+1)/2}, \ldots, -\frac{1}{(n+1)/2}\right)\right\}.$$

One may verify that the lattices $\mathrm{aff}(G_i - p) \cap N_p$ are

$$L_1 = \mathbb{Z}\left(\mathbf{e}_1 - \mathbf{v}_0, \ldots, \mathbf{e}_{(n-1)/2} - \mathbf{v}_0, -\left(\sum_{i=1}^{n-1} \mathbf{e}_i\right) - \mathbf{v}_0\right),$$

and

$$L_2 = \mathbb{Z}(-\mathbf{v}_0, -\mathbf{e}_{(n+1)/2} - \mathbf{v}_0, \ldots, -\mathbf{e}_{n-1} - \mathbf{v}_0) = \mathbb{Z}(\mathbf{v}_0, -\mathbf{e}_{(n+1)/2}, \ldots, -\mathbf{e}_{n-1}).$$

From these descriptions, it is clear that each point of $L = \mathbb{Z}(\mathbf{e}_1, \ldots, \mathbf{e}_{n-1}, \mathbf{v}_0)$ is a sum of a unique element from L_1 and a unique element from L_2.

Therefore, $\overline{F}_0$ is the affine free sum of G_1 and G_2. Since G_2 is a standard simplex, its normalized volume is 1. By Lemma 11, we have

$$\mathrm{NVol}(\overline{F}_0) = \mathrm{NVol}(G_1)\,\mathrm{NVol}(G_2) = \mathrm{NVol}(G_1).$$

Thus it remains to find the normalized volume of G_1, which is unimodularly equivalent to the simplex

$$\mathrm{conv}\{\mathbf{e}_1, \dots, \mathbf{e}_{(n-1)/2}, -(\mathbf{e}_1 + \cdots + \mathbf{e}_{(n-1)/2})\}.$$

It is straightforward to compute (say, by computing the appropriate determinant) that this simplex has a normalized volume of $\frac{n-1}{2} + 1 = \frac{n+1}{2}$, as desired. ∎

This gives us the final piece we need.

Theorem 13. *For a cycle graph of N vertices, the AP bound of (3) is*

$$N\binom{N-1}{\lfloor (N-1)/2 \rfloor}.$$

Proof. We already saw that the conclusion holds when N is odd. When N is even, we recognize that if we can determine the normalized volume of each facet F of P_N, then $\mathrm{conv}\{\mathbf{0} \cup F\}$ is a distinct element of a polytopal decomposition of P_N with the same normalized volume; the latter fact comes from the fact that P_N is totally unimodular, so $\mathrm{conv}\{\mathbf{0} \cup F\}$ is a pyramid over F, which preserves normalized volume.

Since each $\lambda \in \Lambda_n$ corresponds to a unique facet of P_N containing $\mathbf{e}_1 + \cdots + \mathbf{e}_n$, we have

$$|\mathcal{F}(P_N)| = 2\binom{N-1}{N/2-1}.$$

Indeed, there are $\binom{n}{(n-1)/2}$ elements in Λ_n, and this must be doubled to account for the facets containing $-(\mathbf{e}_1 + \cdots + \mathbf{e}_n)$. So, we now simply count

$$\mathrm{NVol}(P_N) = \mathrm{NVol}(\overline{F}_0)|\mathcal{F}(P_N)| = \binom{N}{2} 2\binom{N-1}{N/2-1} = N\binom{N-1}{\lfloor (N-1)/2 \rfloor},$$

as desired. ∎

By the inequalities (5), the AP bound above is also an upper bound for the birationally invariant intersection index. We now show there is no gap between the two. Let $F_{C_N} = (F_{C_N,1}, \dots, F_{C_N,n})$, with each $F_{C_N,i}$ being a generic element from $L_{C_N,i}$, and let $\nabla_{C_N,1}, \dots, \nabla_{C_N,n}$ be their Newton polytopes, respectively.

The BKK bound of the system F_{C_N} coincides with its AP bound by [11, Proposition 1]. We significantly strengthen this result by showing that even though the spaces $L_{C_N,1}, \dots, L_{C_N,n}$ are not generated by monomials, the intersection index $[L_{C_N,1}, \dots, L_{C_N,n}]$ still agrees with the BKK bound for F_{C_N}. This is done by using the strong version of Bernshtein's theorem. We use the following notation: Given a nonzero vector $\mathbf{v} \in \mathbb{R}^n$, $\mathrm{init}_{\mathbf{v}} f$ is defined as $\sum_{\mathbf{a} \in (S)_{\mathbf{v}}} c_{\mathbf{a}} \mathbf{x}^{\mathbf{a}}$, where $(S)_{\mathbf{v}}$ is the subset of S on which the linear functional $\langle \cdot, \mathbf{v} \rangle$ attains its minimum. Extending this notation to a system of Laurent polynomials $F = (f_1, \dots, f_n)$, we write $\mathrm{init}_{\mathbf{v}} F = (\mathrm{init}_{\mathbf{v}} f_1, \dots, \mathrm{init}_{\mathbf{v}} f_n)$.

Theorem 14 ([5, Theorem B]). *Consider a system of n Laurent polynomials $F = (f_1, \dots, f_n)$ in n variables. If $\operatorname{init}_{\mathbf{v}} F$ has no solution in $(\mathbb{C}^*)^n$ for any nonzero vector $\mathbf{v} \in \mathbb{R}^n$, then all solutions of $F(\mathbf{x}) = \mathbf{0}$ are isolated, and the total number is exactly the BKK bound of the system.*

An important fact is that for a generic choice (Remark 1) of the coefficients, the BKK bound is exact.

Lemma 15 ([5]). *Let $F = (f_1, \dots, f_n)$ be a system of n Laurent polynomials in n variables. For generic choices of coefficients, and any nonzero $\mathbf{v} \in \mathbb{R}^n$, $\operatorname{init}_{\mathbf{v}} F$ has no solution in $(\mathbb{C}^*)^n$.*

With these, we show that the birationally invariant intersection index induced by cycle graphs coincides with the AP bound computed in Theorem 13.

Theorem 16. *Given a cycle graph C_N containing $N = n+1$ vertices, let $L_{C_N,1}, \dots, L_{C_N,n}$ be the subspace of rational functions defined in (4). Then*

$$[\, L_{C_N,1}, \dots, L_{C_N,n} \,] = N \binom{N-1}{\lfloor (N-1)/2 \rfloor}.$$

The proof strategy is to show that even though there are algebraic relations among the coefficients for terms in F_{C_N}, such relations will not appear in any nontrivial initial systems.

Proof. Let $\mathbf{v}$ be a vector in $\mathbb{R}^n$. If $(\nabla_{C_N,i})_{\mathbf{v}}$ is a singleton for any $i \in \{1, \dots, n\}$, then $\operatorname{init}_{\mathbf{v}}(F_{\nabla_{C_N,i}})$ has only one term. Consequently the initial system $\operatorname{init}_{\mathbf{v}}(F_{\nabla_{C_N}})$ has no solution in $(\mathbb{C}^*)^n$. It remains to show that the same is true if $(\nabla_{C_N,i})_{\mathbf{v}}$ is not a singleton for any $i = 1, \dots, n$. Since the polytopes $\nabla_{C_N,i}$ all contain the origin, we must have

$$\operatorname{ht}_{\mathbf{v}}(\nabla_{C_N,i}) := \min\{\langle \mathbf{x}, \mathbf{v} \rangle \mid \mathbf{x} \in \nabla_{C_N,i}\} \leq 0 \quad \forall i.$$

If $\operatorname{ht}_{\mathbf{v}}(\nabla_{C_N,i}) = 0$ for all i, then $\langle \pm(\mathbf{e}_i - \mathbf{e}_j), \mathbf{v} \rangle = 0$ for any pair of $(i,j) \in E(C_N)$. It is then easy to verify that $\mathbf{v} = \mathbf{0}$.

Now, supposing $\mathbf{v} \neq \mathbf{0}$, there must be a vertex $i \in \{1, \dots, n\}$ for which $\operatorname{ht}_{\mathbf{v}}(\nabla_{C_N,i}) < 0$. Recall that $\nabla_{C_N,i}$ has at most four vertices: $\{\pm(\mathbf{e}_i - \mathbf{e}_j), \pm(\mathbf{e}_k - \mathbf{e}_j)\}$ where $\{j,k\} = \mathcal{N}_{C_N}(i)$. But the linear functional $\langle \bullet, \mathbf{v} \rangle$ must attain negative values for at least two points in this set. That means there are exactly two points $\mathbf{b}_j \in \{\mathbf{e}_i - \mathbf{e}_j, \mathbf{e}_j - \mathbf{e}_i\}$ and $\mathbf{b}_k \in \{\mathbf{e}_i - \mathbf{e}_k, \mathbf{e}_k - \mathbf{e}_i\}$ such that $\langle \mathbf{b}_j, \mathbf{v} \rangle < 0$ and $\langle \mathbf{b}_k, \mathbf{v} \rangle < 0$. However, since $\mathbf{b}_j \in \nabla_{C_N,j}$ and $\mathbf{b}_k \in \nabla_{C_N,k}$, both $\operatorname{ht}_{\mathbf{v}}(\nabla_{C_N,j})$ and $\operatorname{ht}_{\mathbf{v}}(\nabla_{C_N,k})$ are negative. In other words, if $\operatorname{ht}_{\mathbf{v}}(\nabla_{C_N,i}) < 0$ for some vertex i, then $\operatorname{ht}_{\mathbf{v}}(\nabla_{C_N,j}) < 0$ for any $j \in \mathcal{N}_{C_N}(i)$. Since C_N is connected, as this implication propagates through the graph we can conclude that $\operatorname{ht}_{\mathbf{v}}(\nabla_{C_N,j}) < 0$ for all $i \in \{1, \dots, n\}$. Consequently, for each $(i,j) \in E(C_N)$, the two points $\mathbf{e}_i - \mathbf{e_j}$ and $\mathbf{e}_j - \mathbf{e}_i$ cannot both be in $(\nabla_{C_N,i})_{\mathbf{v}}$ or $(\nabla_{C_N,j})_{\mathbf{v}}$. Recall that $\mathbf{e}_i - \mathbf{e}_j$ and $\mathbf{e}_j - \mathbf{e}_i$ are the exponent vectors of $\frac{x_i}{x_j}$ and $\frac{x_j}{x_i}$, respectively. Therefore either $\frac{x_i}{x_j}$ or $\frac{x_j}{x_i}$ appears in $\operatorname{init}_{\mathbf{v}} F_{C_N}$. Consequently, monomials appearing in $\operatorname{init}_{\mathbf{v}} F_{C_N}$ all have independent coefficients. Then by Lemma 15, for a generic choice of coefficients, the initial system $\operatorname{init}_{\mathbf{v}}(F_{C_N}) = \mathbf{0}$ has no solution in $(\mathbb{C}^*)^n$. This is true for any nonzero vector $\mathbf{v}$, so by Theorem 14 the number of solutions that $F_{C_N} = \mathbf{0}$ has in $(\mathbb{C}^*)^n$ is exactly the BKK bound. Since F_{C_N} is a generic choice, we can conclude that $[L_{C_N,1}, \dots, L_{C_N,n}]$ agrees with the BKK bound and hence the AP bound shown above. ∎

7. Conclusion. Synchronization of coupled oscillators is a fascinating phenomenon with a wide variety of applications in, e.g., living organisms, power grids, and computer and social networks [1, 47]. In this paper, we focused on one of the most popular mathematical models for studying synchronization from an algebraic point of view, namely, the Kuramoto model. In particular, we concentrated on the root counting problem for the synchronization equations, which correspond to the counting of steady states for the corresponding system of ordinary differential equations (2) of the model.

After reformulating the synchronization equations as a system of Laurent polynomials (polynomials that allow negative exponents), we apply the theory of the birationally invariant intersection index to the root counting problem. In particular, we show that the maximum complex root count coincides with the birationally invariant intersection index of a family of rational functions which has rich combinatorial structure over the algebraic torus $(\mathbb{C}^*)^n = (\mathbb{C} \setminus \{0\})^n$. Through this connection and the construction of *adjacency polytopes* (APs), we establish that the upper bounds for the root counts of the Kuramoto equations induced by tree and cycle graphs are 2^{N-1} and $N\binom{N-1}{\lfloor (N-1)/2 \rfloor}$, respectively, where N is the number of vertices in the graph (i.e., the number of oscillators in the network). Both of these bounds are asymptotically tighter than the best known upper bound, which is $\binom{2N-2}{N-1}$. Moreover, our constructive proofs also demonstrated that these two upper bounds are sharp if complex roots are considered. In the case of tree graphs, our upper bound also coincides with a well-known lower bound for maximum real root count. These results confirm a conjecture proposed by a series of recent works [13, 36]: A topology-dependent upper bound for the root count of the Kuramoto system exists and may be much smaller than the general upper bound that depends only on the size of the network.

The explicit formulae of these upper bounds (i.e., the birationally invariant intersection index) established here also have direct practical implications in the effort to numerically solve Kuramoto equations. In the general problem of numerically solving large-scale nonlinear systems of equations, *numerical homotopy methods* have proven to be robust, efficient, and highly scalable [46]. A homotopy construction starts with an upper bound on the number of complex roots that a nonlinear system could have. Then a "target" system is continuously deformed into a "starting" system with the same root count but which can be solved easily (or with known roots). Numerical "continuation" methods are then applied to track the roots under this deformation and thereby solve the target system. In this paper, we provided the first ingredient in this recipe. Moreover, the constructive approach we adopted in the computation of the volume of the APs also produced decompositions or subdivisions that can be constructed algorithmically. It is thus plausible that these will lead to specialized versions of polyhedral homotopy [24] that can be started without the computationally intensive step of "mixed cells computation." This direction will be the focus of our future research. More generally, the upper bounds developed here also serve as explicit stopping criteria for other iterative solvers.

Finally, we point out the interesting parallel between this paper and the recent work by Coss et al. [15] where a different root count is established under the condition that the coupling strength matrix $A = [a_{ij}]$ is of rank 1. Here, A is the weighted adjacency matrix of the underlying graph, and thus the graph topology is reflected in the sparsity pattern

of A. Since both sparsity pattern and rank can be loosely interpreted as measures for the amount of information a matrix actually contains, both works seem to suggest a strong link between the much lower root count (when compared to the most general upper bound) and the informational content of A. It is thus reasonable to ask whether a rigorous link in this direction can be established, thereby unifying both points of view.

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