On the typical and atypical solutions to the Kuramoto equations*

Tianran Chen†, Evgeniia Korchevskaia‡, and Julia Lindberg§

Abstract. The Kuramoto model is a dynamical system that models the interaction of coupled oscillators. There has been much work to effectively bound the number of equilibria to the Kuramoto model for a given network. By formulating the Kuramoto equations as a system of algebraic equations, we first relate the complex root count of the Kuramoto equations to the combinatorics of the underlying network by showing that the complex root count is generically equal to the normalized volume of the corresponding adjacency polytope of the network. We then give explicit algebraic conditions under which this bound is strict and show that there are conditions where the Kuramoto equations have infinitely many equilibria.

Key words. Kuramoto model, adjacency polytope, Bernshtein-Kushnirenko-Khovanskii bound

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1. Introduction.

The Kuramoto model [24] is a simple mathematical model that describes the dynamics on networks of oscillators. It has found applications in neuroscience, biology, chemistry and power systems [4, 18, 20, 31]. Despite its simplicity, it exhibits interesting emergent behaviors. Of particular interest is the phenomenon of frequency synchronization which is when oscillators spontaneously synchronize to a common frequency. In a rotational frame, a synchronization configuration is an equilibrium of the dynamical system, i.e., a root of the Kuramoto equations. This paper aims to understand the structure of these roots in "typical" and "atypical" networks.

Earlier studies focused on statistical analysis of infinite networks [24] but more recently, tools from differential and algebraic geometry have enabled analysis of synchronization on finite networks. For a finite network, knowing the total number of synchronization configurations is fundamental to our understanding of this model. From a computational perspective, this knowledge also plays a critical role in developing numerical methods for finding synchronization configurations. For instance, this number serves as a stopping criterion for *monodromy* algorithms [27] and allows for the development of specialized *homotopy* algorithms [9, 11] for finding synchronization configurations.

In 1982, Ballieul and Byrnes introduced root counting techniques from algebraic geometry to this field and showed that a Kuramoto network of N oscillators has at most $\binom{2N-2}{N-1}$ synchronization configurations [2]. Algebraic geometers will recognize this bound as the bi-homogeneous Bézout bound for an algebraic version of the defining equations. This upper bound can be reached when the network is complete and complex roots (i.e., complex relaxation of the configurations) are counted. However, for sparse networks, the root count (even counting complex roots) can be significantly lower than this upper bound [19, 29], demonstrating the need for a network-dependent root count.

Guo and Salam initiated one of the first algebraic analyses on such sparsity-dependent root counts [19]. Molzahn, Mehta, and Niemerg provided computational evidence for the connection between this root count and network topology [29]. In the special case of rank-one coupling, Coss, Hauenstein, Hong and Molzahn proved this complex root count to be $2^N - 2$, which is also an asymptotically sharp bound on the real root count [16]. Chen, Davis and Mehta established the maximum complex root count for cycle networks to be $N\binom{N-1}{\lfloor (N-1)/2 \rfloor}$ [13], which is sharp as it can be reached with generic choices of coefficients, and it is asymptotically smaller than the bi-homogeneous

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†Department of Mathematics, Auburn University at Montgomery, Montgomery, AL (ti@nranchen.org)

‡School of Mathematics, Georgia Institute of Technology

§Max Planck Institute for Mathematics in the Sciences, Leipzig, Germany (julia.lindberg@mis.mpg.de)

Bézout bound discovered by Baillieul and Byrnes, proving that sparse networks have significantly fewer synchronization configurations than dense ones. Interestingly, Lindberg, Zachariah, Boston, and Lesieutre showed that this bound is attainable by real roots [28]. This bound is an instance of the "adjacency polytope bound" [10]. However, the following was still unknown.

Question 1.1. *For generic choices of coupling coefficients and natural frequencies, will the complex root count for the algebraic Kuramoto equations reach the adjacency polytope bound for all networks? If true, this system is Bernshtein-general, despite the constraints on the coefficients.*

In the first part of this paper, we provide a positive answer to this question and thus establish the generic root count for the Kuramoto equations derived from a graph G to be the normalized volume of the adjacency polytope $\operatorname{conv}\{\vec{e}_i - \vec{e}_j\}_{\{i,j\}\in\mathcal{E}(G)}$. We note that this result is similar to recent work in [5] which shows that the number of (approximate) complex solutions to the Duffing equations is generically the volume of the *Oscillator polytope*. We also extend this generic root count result to variations of the Kuramoto equations.

Even though the generic behavior holds for almost all parameters, understanding the two types of non-generic behavior is still of interest. First, the roots of the Kuramoto equations may remain isolated but the total number can drop below the generic root count. The second part of this paper considers this situation and aims to answer the following.

Question 1.2. *What are the conditions on the coupling coefficients under which the complex roots count for Kuramoto equations will drop below the generic root count?*

Furthermore, for certain networks, we answer the following.

Question 1.3. *What is the gap between the generic root count and actual root count?*

For the second type of non-generic behavior, non-isolated solutions may appear. Ashwin, Bick, and Burylko analyzed certain types of non-isolated solutions in complete networks of identical oscillators [1]. A concrete example of this in a network of four identical oscillators with uniform coupling coefficients was described in [16, Example 2.1]. The existence of non-isolated equilibria for cycle networks was discovered by Lindberg, Zachariah, Boston and Lesieutre [28]. Recent work by Sclosa shows that for every $d \geq 1$ there is a Kuramoto network whose stable equilibria form a manifold of dimension d [32]. In the last part, we aim to answer the following.

Question 1.4. *What are the conditions on the coupling coefficients under which there exist non-isolated synchronization configurations? What are these configurations?*

The rest of this paper is structured as follows. In Section 2, we review concepts that will be used throughout this paper. We then consider the generic root count of the Kuramoto equations in Section 3 and provide an answer to Question 1.1. Next, we turn our attention to non-generic coupling coefficients. In Section 4 we consider when all solutions are isolated and the number of complex solutions drops, answering Questions 1.2 and 1.3. In Section 5 we identify parameter values under which the Kuramoto equations have positive dimensional components, answering Question 1.4. Finally, we conclude with a few remarks in Section 6.

2. Notation and preliminaries. Column vectors, representing points of the lattice $L \cong \mathbb{Z}^n$, are denoted by lowercase letters with arrowhead, e.g., $\vec{a}$. We use boldface letters, e.g., $\boldsymbol{x}$, for points in $\mathbb{C}^n$, $\mathbb{R}^n$, or the dual lattice $L^\vee \cong \mathbb{Z}^n$, and they are written as row vectors. For $\boldsymbol{x} = (x_1, \ldots, x_n)$ and $\vec{a} = (a_1, \ldots, a_n)^\top \in \mathbb{Z}^n$, $\boldsymbol{x}^{\vec{a}} = x_1^{a_1} \cdots x_n^{a_n}$ is a *Laurent monomial* in $\boldsymbol{x}$ with the convention that $x^0 = 1$ (including $x = 0$). A *Laurent polynomial*, f, is a linear combination of Laurent monomials and its *support* and *Newton polytope* are denoted $\operatorname{supp}(f)$ and $\operatorname{Newt}(f)$, respectively. With respect to a vector $\boldsymbol{v}$, the *initial form* of f is the Laurent polynomial $\operatorname{init}_{\boldsymbol{v}}(f)(\boldsymbol{x}) := \sum_{\vec{a}\in(S)_{\vec{v}}} c_{\vec{a}}\, \boldsymbol{x}^{\vec{a}}$, where $(S)_{\vec{v}}$ is the subset of S on which $\langle \boldsymbol{v}\,, \cdot \rangle$ is minimized. For an integer matrix $A = [\vec{a}_1 \ \cdots \ \vec{a}_m]$, $\boldsymbol{x}^A := (\boldsymbol{x}^{\vec{a}_1}, \ldots, \boldsymbol{x}^{\vec{a}_m})$ is a system of Laurent monomials, and we restrict the domain to the *algebraic torus* $(\mathbb{C}^*)^n = (\mathbb{C}\setminus\{0\})^n$ whose group structure is given by $(x_1, \ldots, x_n) \circ (y_1, \ldots, y_n) := (x_1 y_1, \ldots, x_n y_n)$.

We fix G to be a connected graph with vertex set $\mathcal{V}(G) = \{0, 1, \ldots, n\}$ and edge set $\mathcal{E}(G) \neq \varnothing$. For node i of G, $\mathcal{N}_G(i)$ is the set of its adjacent nodes. Arrowheads will be used to distinguish digraphs from graphs, e.g., $\vec{G}$. A more thorough list of notation is included in Appendix A.

2.1. The Kuramoto model. A network of $n+1$ coupled oscillators can be modeled by a graph G with nodes $\mathcal{V}(G) = \{0, \ldots, n\}$ and edges $\mathcal{E}(G)$ representing the oscillators and their connections, respectively. Oscillators have natural frequencies $\vec{w} = (w_0, w_1, \ldots, w_n)^\top$, and, along the edges in G, nonzero constants $K = \{k_{ij}\}$ with $k_{ij} = k_{ji}$ quantify the coupling strength. The structure $(G, K, \vec{w})$ encoding this model will simply be called a **network**. The dynamics are described by

$$\frac{d\theta_i}{dt} = w_i - \sum_{j \in \mathcal{N}_G(i)} k_{ij} \sin(\theta_i - \theta_j) \quad \text{for } i = 0, \ldots, n, \tag{2.1}$$

where θ_i is the phase angle of the i-th oscillator [24]. *Frequency synchronization configurations* are defined to be configurations of $(\theta_0, \ldots, \theta_n)$ for which all oscillators are tuned to have the exact same angular velocity — the average of the natural frequencies $\overline{w}$. By adopting a rotational frame, we can assume $\theta_0 = 0$. Since $k_{ij} = k_{ji}$, we can also eliminate one equation. Therefore, the (frequency) synchronization configurations are the zeroes to the system of n transcendental functions:

$$(w_i - \overline{w}) - \sum_{j \in \mathcal{N}_G(i)} k_{ij} \sin(\theta_i - \theta_j) \quad \text{for } i = 1, \ldots, n. \tag{2.2}$$

The central question of finding the maximum number of synchronization configurations of a Kuramoto network is therefore equivalent to the root counting question for this system.

2.2. Algebraic Kuramoto equations. To leverage the power of root counting techniques from algebraic geometry, the above transcendental system can be reformulated into an algebraic system via the change of variables $x_i = e^{\mathrm{i}\theta_i}$ where $\mathrm{i} = \sqrt{-1}$. Then $\sin(\theta_i - \theta_j) = \frac{1}{2\mathrm{i}}(\frac{x_i}{x_j} - \frac{x_j}{x_i})$, and (2.2) is transformed into a system of n Laurent polynomials $\vec{f}_G = (f_{G,1}, \ldots, f_{G,n})^\top$, given by

$$f_{G,i}(x_1, \ldots, x_n) = \overline{w}_i - \sum_{j \in \mathcal{N}_G(i)} a_{ij} \left(\frac{x_i}{x_j} - \frac{x_j}{x_i} \right) \quad \text{for } i = 1, \ldots, n, \tag{2.3}$$

where $a_{ji} = a_{ij} = \frac{k_{ij}}{2\mathrm{i}}$, and $\overline{w}_i = w_i - \overline{w}$. This system will be referred to as the **algebraic Kuramoto system**,[1] and it captures all synchronization configurations in the sense that the real zeros to (2.2) correspond to the complex zeros of (2.3) on the real torus $(S^1)^n$ (i.e., $|x_i| = |e^{\mathrm{i}\theta}| = 1$).

In much of this paper, we relax the root-counting problem by considering all $\mathbb{C}^*$-zeros of (2.3).[2] One important observation is that we can ignore pendant nodes G in the sense that synchronization configurations naturally extend to pendant nodes.

Lemma 2.1. *[26, Theorem 2.5.1] Suppose $v \neq 0$ is a pendant node of G. Let $G' = G - \{v\}$. Then any $\mathbb{C}^*$-zero of $\vec{f}_{G'}$ extends to two distinct $\mathbb{C}^*$-zeros for $\vec{f}_G$.*

2.3. Power flow equations. The *PV-type power flow system* is one important variation of the Kuramoto system. In it, the graph G models an electric power network where $\mathcal{V}(G) = \{0, \ldots, n\}$ represent buses in the power network. An edge $\{i, j\} \in \mathcal{E}(G)$, representing the connection between

[1] f_G depends on the choice of coupling coefficients $K = [k_{ij}]$ and the natural frequencies $\vec{w}$, and we will use the notation $f_{(G,K)}$ or $f_{(G,K,\vec{w})}$ to emphasize these dependencies, when needed.

[2] Throughout this paper, we talk about $\mathbb{C}^*$-zeros of (2.3), even though there is no difference between $\mathbb{C}^*$-zeros and $\mathbb{C}$-zeros. We emphasize this distinction because in recent algebraic studies of Kuramoto equations, other algebraic formulations of (2.2) have been used, and the phrase "complex zeros/roots of the Kuramoto equations" has a broader meaning, which include zeros at "toric infinity" in the formulation used in this paper.

buses i and j, has a known complex admittance $b'_{ij} + \mathrm{i}g'_{ij}$. For each bus i, the relationship between its complex power injection $P_i + \mathrm{i}Q_i$ and the complex voltages is captured by the nonlinear equations

$$P_i = \sum_{j \in \mathcal{N}_G(i)} |V_i||V_j|(g'_{ij}\cos(\theta_i - \theta_j) + b'_{ij}\sin(\theta_i - \theta_j)) \tag{2.4}$$

$$Q_i = \sum_{j \in \mathcal{N}_G(i)} |V_i||V_j|(g'_{ij}\sin(\theta_i - \theta_j) - b'_{ij}\cos(\theta_i - \theta_j)) \tag{2.5}$$

where V_i and θ_i are the magnitude and the phase angle of the complex voltage at bus i. We fix bus 0 to be the *slack bus*, meaning $\theta_0 = 0$. Equations (2.4)-(2.5) are the *power flow equations*.

The case where Q_i and θ_i are unknown while P_i and $|V_i|$ are known constants are called PV nodes and they model typical generator buses. As above, through the change of variables $x_i = e^{\mathrm{i}\theta_i}$ we get the *PV-type algebraic power flow equations*

$$f_{G,i}(x_1, \ldots, x_n) = P_i - \sum_{j \in \mathcal{N}_G(i)} g_{ij}\left(\frac{x_i}{x_j} + \frac{x_j}{x_i}\right) + b_{ij}\left(\frac{x_i}{x_j} - \frac{x_j}{x_i}\right), \quad \text{for } i = 1, \ldots, n \tag{2.6}$$

where $b_{ij} = \frac{1}{2\mathrm{i}}|V_i||V_j|b'_{ij}$ and $g_{ij} = \frac{1}{2}|V_i||V_j|g'_{ij}$. When $g_{ij} = 0$, the corresponding power system is *lossless* and (2.6) reduces to (2.3). Otherwise, the system is *lossy* and (2.6) differs from (2.3) in the constraints on the coefficients. Yet, the same set of monomials are involved, and as we will demonstrate, the algebraic arguments we will develop can be applied to this generalization.

2.4. Kuramoto equations with phase delays. Another generalization of the Kuramoto model considers *phase delays*. With known phase delay parameters $\delta_{ij} \in \mathbb{R}$, (2.2) is augmented to:

$$0 = w_i - \overline{w} - \sum_{j \in \mathcal{N}_G(i)} k_{ij}\sin(\theta_i - \theta_j + \delta_{ij}), \quad \text{for } i = 1, \ldots, n,$$

which corresponds to the system in which for a pair of coupled oscillators i and j, oscillator i responds not directly to the phase angle of oscillator j but its delayed phase $\theta_j - \delta_{ij}$. Letting $C_{ij} = e^{\mathbf{i}\delta_{ij}}$, we can again make this system algebraic giving:

$$f_{G,i}(x_1, \ldots, x_n) = \overline{w}_i - \sum_{j \in \mathcal{N}_G(i)} a_{ij}\left(\frac{x_i C_{ij}}{x_j} - \frac{x_j}{x_i C_{ij}}\right), \quad \text{for } i = 1, \ldots, n. \tag{2.7}$$

This is, again, a generalization of the algebraic Kuramoto system (2.3) that involves the same set of monomials. The algebraic results to be developed also apply to this family of algebraic systems.

2.5. BKK bound. The root counting arguments in this paper revolve around the Bernshtein-Kushnirenko-Khovanskii (BKK) bound, especially Bernshtein's Second Theorem.

Theorem 2.2 (D. Bernshtein 1975 [3]). *(A) For a square Laurent system $\vec{f} = (f_1, \ldots, f_n)$, if for all nonzero vectors $\vec{v} \in \mathbb{R}^n$, $\mathrm{init}_{\vec{v}}(\vec{f})$ has no $\mathbb{C}^*$-zeros, then all $\mathbb{C}^*$-zeros of $\vec{f}$ are isolated, and the total number, counting multiplicity, is the mixed volume $M = \mathrm{MV}(\mathrm{Newt}(f_1), \ldots, \mathrm{Newt}(f_n))$.*

(B) If $\mathrm{init}_{\vec{v}}(\vec{f})$ has a $\mathbb{C}^*$-zero for some $\vec{v} \neq \vec{0}$, then the number of isolated $\mathbb{C}^*$-zeros $\vec{f}$ has, counting multiplicity, is strictly less than M if $M > 0$.

A system for which the condition (A) holds is said to be *Bernshtein-general*. Only the special case of identical Newton polytopes, i.e., when $\mathrm{Newt}(f_1), \ldots, \mathrm{Newt}(f_n)$ are all identical, will be used. This specialized version strengthens Kushnirenko's Theorem [25].

2.6. Randomized algebraic Kuramoto system. The analysis of the algebraic Kuramoto system (2.3) can be further simplified through "randomization" (a.k.a. "pre-conditioning"). Taking $\vec{f}_G$ as a column vector, for any nonsingular matrix R, the systems $\vec{f}_G$ and $\vec{f}^*_G := R\,\vec{f}_G$ have the

same zero set. With a generic choice of R, there will be no complete cancellation of terms, and $\vec{f}^*_G$ will be referred to as the **randomized (algebraic) Kuramoto system**. This system is *unmixed* in the sense that for all $i \in [n]$, $f^*_{G,i}$ have identical supports since they involve the same set of monomials, namely, constant terms, $x_i x_j^{-1}$ and $x_j x_i^{-1}$ for every edge $\{i,j\} \in \mathcal{E}(G)$.

The randomization $\vec{f}_G \mapsto \vec{f}^*_G$ does not alter the zero set but makes a very helpful change to the tropical structure in the sense that much is already known about the tropical self-intersection of $\vec{f}^*_G$. In particular, there is a mapping between the graph-theoretical features of G and the tropical structures of $\vec{f}^*_G$ through which we can gain key insight into the structure of the zeros of $\vec{f}^*_G$. The next three subsections briefly review the geometric interpretation of this connection.

2.7. Adjacency polytopes. For each $i = 1, \ldots, n$, $\operatorname{supp}(f^*_{G,i})$ are all identical and given by

$$\check{\nabla}_G := \{\pm(\vec{e}_i - \vec{e}_j) \mid \{i,j\} \in \mathcal{E}(G)\} \cup \{\vec{0}\},$$

where $\vec{e}_i$ is the i-th standard basis vector of $\mathbb{R}^n$ for $i = 1, \ldots, n$, and $\vec{e}_0 = \vec{0}$. In Refs. [10, 13], the convex hull of $\check{\nabla}_G$ is called the *adjacency polytope* of G (of PV-type[3]). It is unimodularly equivalent to the *symmetric edge polytope* of G that has appeared in number theory and discrete geometry (see, for example, the broad overview provided in [17]).

We will not distinguish $\check{\nabla}_G$ from its convex hull, meaning a "face" of $\check{\nabla}_G$ refers to a subset $F \subseteq \check{\nabla}_G$ such that $\operatorname{conv}(F)$ is a face of $\operatorname{conv}(\check{\nabla}_G)$ and $\dim(F) := \dim(\operatorname{conv}(F))$. The set of facets and the boundary of $\check{\nabla}_G$ are denoted $\mathcal{F}(\check{\nabla}_G)$ and $\partial\check{\nabla}_G$, respectively. The *corank*[4] of a face F, denoted $\operatorname{corank}(F)$, is the number $|F| - \dim(F) - 1$.

By Kushnirenko's Theorem [25], the *normalized volume* of the adjacency polytope, denoted $\operatorname{Vol}(\check{\nabla}_G)$, is an upper bound for the $\mathbb{C}^*$-zero count for $\vec{f}^*_G$ and $\vec{f}_G$ which also bounds the real zero count to the transcendental Kuramoto system (2.2). This is the *adjacency polytope bound* [10, 13].

2.8. Faces and face subgraphs. There is an intimate connection between faces of $\check{\nabla}_G$ and subgraphs of G. Since $\vec{0}$ is an interior point of $\check{\nabla}_G$, every vertex of a proper face F of $\check{\nabla}_G$ is of the form $\vec{e}_i - \vec{e}_j$ for some $\{i,j\} \in \mathcal{E}(G)$. Thus, it is natural to consider the corresponding *facial subgraph* G_F and *facial subdigraph* $\vec{G}_F$, given by

$$\mathcal{E}(G_F) = \{\{i,j\} \mid \vec{e}_i - \vec{e}_j \in F \text{ or } \vec{e}_j - \vec{e}_i \in F\} \text{ and}$$
$$\mathcal{E}(\vec{G}_F) = \{(i,j) \mid \vec{e}_i - \vec{e}_j \in F\}.$$

As defined in [9, 17], for $F \in \mathcal{F}(\check{\nabla}_G)$, $\mathcal{E}(G_F)$ is called a *facet subgraph* and $\mathcal{E}(\vec{G}_F)$ a *facet subdigraph*. Higashitani, Jochemko, and Michałek provided a topological classification of face subgraphs [22, Theorem 3.1] and it was later reinterpreted [12, Theorem 3]. We state the latter here.

Theorem 2.3 (Theorem 3 [12]). *Let H be a nontrivial connected subgraph of G.*

1. *H is a face subgraph of G if and only if it is a maximal bipartite subgraph of $G[\mathcal{V}(H)]$.*
2. *H is a facet subgraph of G if and only if it is a maximal bipartite subgraph of G.*

Multiple faces can correspond to the same facial subgraph (in particular, $G_F = G_{(-F)}$ for any face F). The crisper parametrization is given by the correspondence $F \mapsto \vec{G}_F$. Ref. [12] describes the balancing conditions that characterize facial subdigraphs. Here, we only make use of the balancing condition for the "primitive cycle vectors" associated with a face subdigraph.

For a facial subdigraph $\vec{G}_F$, its *reduced incidence matrix* $\check{Q}(\vec{G}_F)$ is the matrix with columns $\vec{e}_i - \vec{e}_j$ such that $(i,j) \in \mathcal{E}(\vec{G}_F)$ (see Appendix A). For a face F of $\check{\nabla}_G$ such that G_F is connected,

[3] This is retroactively named the adjacency polytope of PV-type to distinguish it from those of PQ-type [15]. In general, an adjacency polytope associated with a graph G is the convex hull of $\{\vec{0}\}$ and point sets $\{P_{ij} \mid \{i,j\} \in \mathcal{E}(G)\}$ that depend on the "types" of the nodes, a concept originally derived from the power-flow studies (Subsection 2.3).

[4] In matroid theory, the more common term is "*nullity*". Here, we follow the convention from convex geometry.

its corank d equals the nullity of $\check{Q}(\vec{G}_F)$. The null space is spanned by d vectors with entries $\{+1, 0, -1\}$, indicating the incidence of arcs with the cycles in G_F with prescribed orientations. These vectors will be referred to as *primitive cycle vectors*.

Lemma 2.4 (Theorem 7 [12]). *Let F be a face of $\check{\nabla}_G$ for which G_F is connected. For any cycle in G_F, let $\vec{\eta}$ be its associated primitive cycle vector with respect to $\vec{G}_F$, then $\langle \mathbf{1}, \vec{\eta} \rangle = 0$.*

Note that these conditions are only necessary conditions on the primitive cycle vectors. If G contains more than one cycle, these conditions are not sufficient. Stronger classifications results (which are not needed here) can be found in [12, 22].

For a subdigraph $\vec{H}$ of $\vec{G}$, the *coupling vector* $\boldsymbol{k}(\vec{H})$ has entries k_{ij} for $(i,j) \in \vec{H}$. Similarly, the entries of $\boldsymbol{a}(\vec{H})$ are $a_{ij} = \frac{k_{ij}}{2\mathrm{i}}$ for $(i,j) \in \mathcal{E}(\vec{H})$. The ordering of the entries is arbitrary, but when appearing in the same context with $\check{Q}(\vec{H})$, consistent ordering is implied.

2.9. Facial systems. The vast literature on the facial structure of $\check{\nabla}_G$ gives us a shortcut to understanding the initial systems of the randomized algebraic Kuramoto system $\vec{f}^*_G$. Recall that $\operatorname{supp}(f^*_{G,i})$ are all identical, so the initial systems of $\vec{f}^*_G$ have particularly simple descriptions corresponding to proper faces of $\check{\nabla}_G$. For any $0 \neq \boldsymbol{v} \in \mathbb{R}^n$, the initial system $\operatorname{init}_{\boldsymbol{v}}(\vec{f}^*_G)$ is

$$\operatorname{init}_{\boldsymbol{v}}(f^*_{G,i})(\boldsymbol{x}) = \sum_{\vec{e}_j - \vec{e}_{j'} \in F} c_{i,j,j'}\, \boldsymbol{x}^{\vec{e}_j - \vec{e}_{j'}} = \sum_{(j,j') \in \mathcal{E}(\vec{G}_F)} c_{i,j,j'}\, \boldsymbol{x}^{\vec{e}_j - \vec{e}_{j'}} \quad \text{for } i = 1, \ldots, n, \tag{2.8}$$

where F is the proper face of $\check{\nabla}_G$ for which $\boldsymbol{v}$ is an inner normal vector. We will make frequent use of this geometric interpretation and therefore it is convenient to slightly abuse the notation and write $\operatorname{init}_F(\vec{f}^*_G) := \operatorname{init}_{\vec{v}}(\vec{f}^*_G)$ when the particular choice of vector $\vec{v}$ defining the face F is not important. It will be called a *facial system* of $\vec{f}^*_G$, (or a *facet system* if $F \in \mathcal{F}(\check{\nabla}_G)$).

3. Generic complex root count. In this section, we establish the $\mathbb{C}^*$-zero count for the algebraic Kuramoto system (2.3) for generic choices of real constants $\overline{w}_1, \ldots, \overline{w}_n$ and $\{k_{ij}\}_{\{i,j\} \in \mathcal{E}(G)}$. We show that the generic $\mathbb{C}^*$-zero count is the adjacency polytope bound $\operatorname{Vol}(\check{\nabla}_G)$. A corollary is that this system is Bernshtein-general, despite the constraints on the coefficients.

The first observation is that since $\mathbb{R}$ is Zariski-dense in $\mathbb{C}$, generic choices of complex parameters can be used without changing the $\mathbb{C}^*$-zero count. That is, it is sufficient to focus on the root counting problem for the system

$$0 = \overline{w}_i - \sum_{j \in \mathcal{N}_G(i)} a_{ij} \left(\frac{x_i}{x_j} - \frac{x_j}{x_i} \right) \qquad \text{for } i = 1, \ldots, n,$$

in the n unknowns $x_1, \ldots, x_n$ (with $x_0 = 1$) for generic *complex* choices[5] of $\overline{w}_i, a_{ij}$.

There are three main obstacles. First, in each polynomial the monomials x_i/x_j and x_j/x_i share the same coefficient a_{ij}. Second, for any edge $\{i,j\} \in \mathcal{E}(G)$, the i-th and j-th polynomials have the terms $a_{ij}(x_i x_j^{-1} - x_j x_i^{-1})$ and $a_{ji}(x_j x_i^{-1} - x_i x_j^{-1})$, respectively, which are negations of each other since $a_{ij} = a_{ji}$. Therefore, the allowed choices of coefficients consists of a nowhere dense subset of (Lebesgue) measure 0 in the space of all possible complex coefficients. Finally, unless G is the complete graph, the Newton polytopes of the algebraic Kuramoto system are not full-dimensional which prevents simpler arguments (e.g. [8]) from being applied. We will show that despite this, the maximum $\mathbb{C}^*$-zero count, given by the adjacency polytope bound is generically attained.

Before presenting the main theorem of this section, we first establish a few technical results that will be used, some of which are well known, but are nonetheless included for completeness.

[5] For simplicity, we will assume the constant terms $\overline{w}_1, \ldots, \overline{w}_n$ are chosen generically. This is equivalent to choosing $w_1, \ldots, w_n$ generically and leaving w_0 constrained by these choices.

Lemma 3.1. *The lifting function* $\tilde{\omega} : \check{\nabla}_G \to \mathbb{Q}$, *given by*

$$\tilde{\omega}(\vec{x}) = \begin{cases} 0 & \text{if } \vec{x} = \vec{0} \\ 1 & \text{otherwise} \end{cases}$$

induces a regular subdivision $\Sigma_\omega(\check{\nabla}_G) = \{\vec{0} \cup F \mid F \in \mathcal{F}(\check{\nabla}_G)\}$.

This is a well known consequence of assigning a sufficiently small lifting value to an interior point ($\vec{0}$ in this case). The resulting regular subdivision may be too coarse to be useful in our discussions. Indeed, it will not be a triangulation unless G is has no even cycles [12, 11]. It can be refined into a triangulation through perturbations on the nonzero lifting values.

Lemma 3.2. *For generic but symmetric choices* $\{\delta_{ij} = \delta_{ji} \in \mathbb{R} \mid \{i,j\} \in \mathcal{E}(G)\}$ *that are sufficiently close to 0, the function* $\omega : \check{\nabla}_G \to \mathbb{Q}$, *given by*

$$\omega(\vec{a}) = \begin{cases} 0 & \text{if } \vec{a} = \vec{0} \\ 1 + \delta_{ij} & \text{if } \vec{a} = \vec{e}_i - \vec{e}_j \end{cases}$$

induces a regular unimodular triangulation $\Delta_\omega = \Sigma_\omega(\check{\nabla}_G)$ *that is a refinement of* $\Sigma_{\tilde{w}}(\check{\nabla}_G)$ *and its cells are in one-to-one correspondence with cells in a unimodular triangulation of* $\partial\check{\nabla}_G$. *Indeed, each cell is of the form* $0 \cup \Delta$ *where* Δ *is a simplex in* $\partial\check{\nabla}_G$.

Proof. We first show the interior point $\vec{0}$ is contained in every cell. Let $C \in \Delta_\omega$ and $(\vec{v}, 1)$ be the inner normal vector the lower facet of $\check{\nabla}_G^\omega$ whose projection is C. Suppose $\vec{0} \notin C$, then C contains an affinely independent set of $n+1$ points $\{\vec{a}_0, \ldots, \vec{a}_n\} \not\ni \vec{0}$, and $\vec{v}$ satisfies the equation

$$\begin{bmatrix} \vec{a}_1^\top - \vec{a}_0^\top \\ \vdots \\ \vec{a}_n^\top - \vec{a}_0^\top \end{bmatrix} \vec{v} = \begin{bmatrix} \omega(\vec{a}_0) - \omega(\vec{a}_1) \\ \vdots \\ \omega(\vec{a}_0) - \omega(\vec{a}_n) \end{bmatrix}.$$

Let B be the matrix on the left and $\vec{\beta}$ be the vector on the right, then $\vec{v} = B^{-1}\vec{\beta}$ and thus

$$|\langle \vec{a}, \vec{v} \rangle| \le \|\vec{a}\|\|\vec{v}\| \le \|\vec{a}\| \, \|B^{-1}\| \, \|\vec{\beta}\| \quad \text{for any } \vec{a} \in C.$$

Note that entries of $\vec{\beta}$ are differences among the $\{\delta_{ij}\}$. Therefore, for any $\epsilon > 0$, there is a δ such that $\delta_{ij} < \delta$ for all $\{i,j\} \in \mathcal{E}(G)$ implies $|\langle \vec{a}, \vec{v} \rangle| < \epsilon$, contradicting with the assumption that

$$\langle \vec{v}, \vec{a} \rangle + \omega(\vec{a}) < \langle \vec{v}, \vec{0} \rangle + \omega(\vec{0}) = 0.$$

This shows that $\vec{0}$ must be contained in every cell.

To establish Δ_ω as a triangulation, it is sufficient to show nonzero points in a cell $C \in \Delta_\omega$ are assigned independent lifting values by ω. If $\pm(\vec{e}_i - \vec{e}_j) \in C$, for some $\{i,j\} \in \mathcal{E}(G)$, then

$$\langle \vec{v}, \pm(\vec{e}_i - \vec{e}_j) \rangle + \omega(\pm(\vec{e}_i - \vec{e}_j)) = h,$$

where h is the minimum of $\langle \vec{v}, \cdot \rangle + \omega(\cdot)$ over $\check{\nabla}_G$. Since $\omega(\pm(\vec{e}_i - \vec{e}_j)) = 1 + \delta_{ij}$, summing the two equation produces

$$1 + \delta_{ij} = h > 0,$$

which contradict with constraint that $0 = \langle \vec{v}, \vec{0} \rangle + \omega(\vec{0}) \ge h$. Therefore, if $\vec{e}_i - \vec{e}_j \in C$, then $\vec{e}_j - \vec{e}_i \notin C$. Consequently, the nonzero points in C are associated with independent generic choices of lifting values, and thus C must be a simplex.

For unimodularity, note that, since $\vec{e}_i - \vec{e}_j \in C$ implies $\vec{e}_j - \vec{e}_i \notin C$ for $\{i,j\} \in \mathcal{E}(G)$ and for a cell $C \in \Delta_\omega$, the nonzero points of C, as vectors, are exactly columns of (signed) incidence matrix of G, which is unimodular. Therefore, each cell of Δ_ω is unimodular.

Finally, we show that Δ_ω is a refinement of $\Sigma_{\tilde{\omega}}(\check{\nabla})$ from Lemma 3.1. In the following, we shall fix an arbitrary ordering of the points in $\check{\nabla}_G$ and consider lifting functions on $\check{\nabla}_G$, e.g., ω and $\tilde{\omega}$, as vectors in $\mathbb{R}^{|\check{\nabla}_G|}$, whose entries are their lifting values. Let $\mathcal{C} = \mathcal{C}(\check{\nabla}_G, \Delta_\omega)$ be the (closed) secondary cone of Δ_ω in $\check{\nabla}_G$. Then $\mathcal{C}$ is full-dimensional, since Δ_ω is a triangulation. By assumption, ω is sufficiently close to $\tilde{\omega}$, so $\tilde{\omega} \in \mathcal{C}$. Consequently, $\Delta_\omega = \Sigma_\omega(\check{\nabla}_G)$ equals or refines $\Sigma_{\tilde{\omega}}(\check{\nabla}_G)$. ■

A lifting function $\omega : \check{\nabla}_G \to \mathbb{Q}$ for which Lemma 3.2 holds will be referred to as a **generic symmetric lifting function** for $\check{\nabla}_G$.

Lemma 3.3. *For a generic symmetric lifting function ω for $\check{\nabla}_G$, let Δ be a simplex in Δ_ω. Then the digraph $\vec{G}_\Delta$ is acyclic, and its underlying graph G_Δ is a spanning tree of G.*

The proof of this lemma is nearly identical to the proof of [9, Theorem 1] and we include an elementary proof in Appendix B for completeness.

Lemma 3.4. *Suppose $\vec{T} < \vec{G}$ is a acyclic, and its underlying graph T is a spanning tree of G, then, for generic $\overline{w}_1, \dots, \overline{w}_n \in \mathbb{C}^*$ and* any *choices of $a_{ij} \in \mathbb{C}^*$, the system of n Laurent polynomials*

$$\overline{w}_i - \sum_{j \in \mathcal{N}^+_{\vec{T}}(i)} a_{ij} \frac{x_i}{x_j} + \sum_{j \in \mathcal{N}^-_{\vec{T}}(i)} a_{ij} \frac{x_j}{x_i} \quad \text{for } i = 1, \dots, n$$

has a unique zero in $(\mathbb{C}^*)^n$, and this zero is isolated and regular.

Here, $\mathcal{N}^+_{\vec{T}}(i) = \{j \in \mathcal{V}(\vec{T}) \mid (i,j) \in \mathcal{E}(\vec{T})\}$ and $\mathcal{N}^-_{\vec{T}}(i) = \{j \in \mathcal{V}(\vec{T}) \mid (j,i) \in \mathcal{E}(\vec{T})\}$ are the sets of adjacent nodes of i through outgoing and incoming arcs, respectively.

Proof. We prove this by induction on the number of vertices of G, N. Denote the above system as $\vec{f}_{\vec{T}}(x_0, \dots, x_{N-1})$. The statement is true for the case where $N = 2$ and $|\mathcal{V}(G)| = |\mathcal{V}(T)| = 2$.

Assume the statement is true for any graph with N nodes and consider the case G has $N+1$ nodes $\{0, 1, \dots, N\}$. By re-labeling, we can assume node N is a leaf in T and is adjacent to node $N-1$. Moreover, since this system is homogeneous of degree 0, we can also scale the homogeneous coordinates so that $x_{N-1} = 1$. Then the above system is decomposed Laurent system $\vec{f}_{\vec{T}'}(x_0, \dots, x_{N-1})$ and the binomial $\overline{w}_N \pm a_{N,N-1} x_N^{\mp 1}$, where $\vec{T}'$ is the tree $\vec{T} - \{N\}$. The induction hypothesis, that $\vec{f}_{\vec{T}'}(x_0, \dots, x_N)$ has a unique isolated and regular thus completes the proof.

With these technical preparations, we now establish the main theorem of this section.

Theorem 3.5. *For generic choices of real or complex constants $\overline{w}_1, \dots, \overline{w}_n$ and generic but symmetric choices of real or complex coupling coefficients $k_{ij} = k_{ji} \neq 0$ for $\{i,j\} \in \mathcal{E}(G)$, the $\mathbb{C}^*$-zero set of the algebraic Kuramoto system (2.3) $\vec{f}_G = (f_{G,1}, \dots, f_{G,n})$, given by*

$$f_{G,i}(x_1, \dots, x_n) = \overline{w}_i - \sum_{j \in \mathcal{N}_G(i)} \frac{k_{ij}}{2\mathrm{i}} \left(\frac{x_i}{x_j} - \frac{x_j}{x_i} \right) = 0$$

consists of isolated regular points, and the total number is $\mathrm{Vol}(\check{\nabla}_G)$.

This is a far generalization of [13, Corollary 9 and Theorem 16] and [19, Theorem 3.3.3] the generic $\mathbb{C}^*$-zero count is established for tree[6] cycle, and complete networks only.

Proof. By Sard's Theorem, for generic choices of $\overline{w}_1, \dots, \overline{w}_n$, the $\mathbb{C}^*$-zero set of $\vec{f}_G$ consists of isolated and regular points. Under this assumption, we only need to establish the $\mathbb{C}^*$-zero count. Moreover, since it is already known this zero count is bounded by $\mathrm{Vol}(\check{\nabla}_G)$, it is sufficient to show that it is greater than or equal to this bound. We shall take a constructive approach through a specialized version of the polyhedral homotopy of Huber and Sturmfels [23].

[6] The generic number of complex equilibria for the Kuramoto model on tree networks is 2^n. This fact appears to be well known among researchers in power systems long before the referenced paper [13].

Let $\omega : \check{\nabla}_G \to \mathbb{Q}$ be a generic symmetric lifting function for $\check{\nabla}_G$. We use the notation $\omega_{ij} = \omega(\vec{e}_i - \vec{e}_j) = \omega(\vec{e}_j - \vec{e}_i)$ and define the function $\vec{h} = (h_1, \ldots, h_n) : (\mathbb{C}^*)^n \times \mathbb{C} \to \mathbb{C}^n$, given by

$$h_i(x_1, \ldots, x_n, t) = \overline{w}_i - \sum_{j \in \mathcal{N}_G(i)} a_{ij} t^{\omega_{ij}} \left(\frac{x_i}{x_j} - \frac{x_j}{x_i} \right).$$

Away from $t = 0$, $\vec{h}$ is a parametrized version of the original system $\vec{f}_G$ with coefficients being analytic functions of the parameter t. Note that since $\omega_{ij} = \omega_{ji}$, for any choice of $t \in \mathbb{C}$, the system still satisfies the symmetry constraints on the coefficients. By the Parameter Homotopy Theorem [30], for t outside a proper Zariski closed (i.e., finite) set $Q \subset \mathbb{C}$, the number of isolated $\mathbb{C}^*$-zeros of $\vec{h}(\,\cdot\,, t) = 0$ is a constant which is also an upper bound for the isolated $\mathbb{C}^*$-zero count for $\vec{h}(\,\cdot\,, t)$ for all $t \in \mathbb{C}$. Let η be this generic $\mathbb{C}^*$-zero count. Since $\overline{w}_i$ and a_{ij} are chosen generically, and $\vec{h}(\,\cdot\,, 1) \equiv \vec{f}_G(\,\cdot\,)$, we can conclude that $t = 1$ is outside Q and thus η agrees with the generic $\mathbb{C}^*$-zero count of $\vec{f}_G$ that we aims to establish. That is, it is sufficient to show $\eta \geq \mathrm{Vol}(\check{\nabla}_G)$.

Taking a constructive approach, we now construct η smooth curves (of one real-dimension), that will connect the $\mathbb{C}^*$-zeros of $\vec{f}_G$ and the collection of $\mathbb{C}^*$-zeros of the special systems described in Lemma 3.4. This connection allows us to count the number of curves.

Along a ray $t : (0, 1] \to \mathbb{C}$ on the complex plane parametrized by $t(\tau) = \tau e^{i\theta}$ for a choice of $\theta \in [0, 2\pi)$ that avoids Q (i.e., $t(\tau) \notin Q$ for all $\tau \in (0, 1]$), the function $\vec{h}$ is given by

$$h_i(x_1, \ldots, x_n, t) = h_i(x_1, \ldots, x_n, \tau e^{i\theta}) = \overline{w}_i - \sum_{j \in \mathcal{N}_G(i)} a_{ij} e^{i\theta\omega_{ij}} \tau^{\omega_{ij}} \left(\frac{x_i}{x_j} - \frac{x_j}{x_i} \right),$$

and $\vec{h}$ has η isolated and regular $\mathbb{C}^*$-zeros for all $\tau \in (0, 1]$. By the principle of homotopy continuation, the zero set of $\vec{h}$ form η smooth curves in $(\mathbb{C}^*)^n \times (0, 1]$ smoothly parametrizable by τ. The problem is now reduced to counting these curves.

Fixing such a curve C, the asymptotic behavior of C as $\tau \to 0$, in a compactification of $(\mathbb{C}^*)^n$ can be characterized by the solutions to an initial system of $\vec{h}$, as a system of Laurent polynomials in $\mathbb{C}\{\tau\}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. Stated in affine coordinates, by the Fundamental Theorem of Tropical Geometry, there exists a system of convergent Puiseux series $x_1(\tau), \ldots, x_n(\tau) \in \mathbb{C}\{\tau\}$ that represents the germ of C, as an analytic variety at $\tau = 0$, such that the leading coefficients, as a point in $(\mathbb{C}^*)^n$ satisfies the initial system $\mathrm{init}_{\boldsymbol{v}}(\vec{h})$,[7] and $\boldsymbol{v} = (v_1, \ldots, v_n) \in \mathbb{Q}^n$ are the orders of the Puiseux series $x_1(\tau), \ldots, x_n(\tau) \in \mathbb{C}\{\tau\}$. Therefore, it is sufficient to show that there are at least $\mathrm{Vol}(\check{\nabla}_G)$ distinct initial systems of $\vec{h}$ that will each contribute one solution path.

By Lemma 3.2, the regular subdivision $\Delta_\omega = \Sigma_\omega(\check{\nabla}_G)$ is a unimodular triangulation. Fix a simplex $\Delta \in \Delta_\omega$, and let $(\boldsymbol{v}, 1)$ be the upward pointing inner normal vector that defines Δ, then by construction, there is an affinely independent set $\{\vec{a}_1, \ldots, \vec{a}_n\} \subset \partial\check{\nabla}_G$ such that

$$\begin{aligned} \langle\, \boldsymbol{v}\,,\, \vec{a}_k \,\rangle + \omega(\vec{a}_k) &= 0 \quad \text{for } k = 1, \ldots, n, \text{ and} \\ \langle\, \boldsymbol{v}\,,\, \vec{a} \,\rangle + \omega(\vec{a}) &> 0 \quad \text{for all } \vec{a} \in \check{\nabla}_G \setminus \{\vec{0}, \vec{a}_1, \ldots, \vec{a}_n\}. \end{aligned}$$

Therefore, the exponent vectors associated with monomials with nonzero coefficients in $\mathrm{init}_{\boldsymbol{v}}(\vec{h})$ are exactly $\{0, \vec{a}_1, \ldots, \vec{a}_n\}$. By Lemma 3.3, $\vec{T} = \vec{G}_\Delta$ is an acyclic subdigraph and its associated subgraph T is a spanning tree of G. Indeed, $\mathrm{init}_{\boldsymbol{v}}(\vec{h})$ consists of Laurent polynomials

$$\overline{w}_i - \sum_{j \in \mathcal{N}^+_{\vec{T}}(i)} a_{ij} \frac{x_i}{x_j} + \sum_{j \in \mathcal{N}^-_{\vec{T}}(i)} a_{ij} \frac{x_j}{x_i} \quad \text{for } i = 1, \ldots, n.$$

[7] We do not claim $\vec{h}$ form a tropical basis, nor do we require that $\mathrm{init}_{\boldsymbol{v}}(\vec{h})$ generate the corresponding initial ideal. Yet, as we will show, an initial system of $\vec{h}$ of the special form we will describe is sufficient to uniquely determine the leading coefficients of the Puiseux series expansion of a solution path.

By Lemma 3.4, this system has a unique $\mathbb{C}^*$-zero, which is regular. Following from the principle of homotopy continuation, it gives rise to a smooth curve defined by $\vec{h}(\boldsymbol{x}, \tau e^{i\theta}) = 0$ in $(\mathbb{C}^*)^n \times (0, 1]$.

This argument holds for every simplex in Δ_ω. That is, each simplex $\Delta \in \Delta_\omega$ contributes a curve define by $\vec{h} = \vec{0}$. Moreover, these curves are distinct, so the number of curves η satisfies $\eta \geq |\Delta_\omega| = \mathrm{Vol}(\check{\nabla}_G)$, which completes the proof. ∎

This establishes the fact that the generic $\mathbb{C}^*$-zero count for the algebraic Kuramoto system $\vec{f}_G$ is exactly the adjacency polytope bound, despite the algebraic constraints on the parameters. Note that the BKK bound for $\vec{f}_G$ is always between the generic $\mathbb{C}^*$-zero count and the adjacency polytope bound. Therefore, we can indirectly derive the Bernshtein-genericity of $\vec{f}_G$.

Corollary 3.6. *For generic real or complex $\overline{w}_1, \ldots, \overline{w}_n$ and generic but symmetric real or complex $k_{ij} = k_{ji} \neq 0$ for $\{i, j\} \in \mathcal{E}(G)$, the algebraic Kuramoto system (2.3) is Bernshtein-general, and*

$$\mathrm{MV}(\mathrm{Newt}(f_{G,1}), \ldots, \mathrm{Newt}(f_{G,n})) = \mathrm{Vol}(\mathrm{conv}(\mathrm{Newt}(f_{G,1}) \cup \cdots \cup \mathrm{Newt}(f_{G,n}))) = \mathrm{Vol}(\check{\nabla}_G).$$

Remark 3.7. We remark that the proof of Theorem 3.5 utilizes many ideas from the polyhedral homotopy of Huber and Sturmfels [23]. The main differences are that we consider a lifting function that preserves the relationships among the coupling coefficients and we choose generators of the ideals at "toric infinity" with a nice tree structure instead of requiring them to be binomial.

Finally, we draw attention to the tropical nature of this proof. The first half of the proof establishes the tropical version of the generic root count result: With the assignment of the valuation $\mathrm{val}(\overline{w}_i) = 0$ and generic but symmetric $\mathrm{val}(a_{ij}) = \mathrm{val}(a_{ji}) = \omega_{ij} = \omega_{ji}$, we showed that the intersection number of the tropical hypersurfaces defined by $f_{G,1}, \ldots, f_{G,n}$ is bounded below by $\mathrm{Vol}(\check{\nabla})$ which is also the stable self-intersection number of the tropical hypersurface defined by the randomized Kuramoto system (as defined in Subsection 2.6) with the same valuation. Computing generic root count via stable tropical intersection is the topic of a recent paper by Paul Helminck and Yue Ren [21], in which the special case for complete networks is studied as an example.

Another consequence of Theorem 3.5 is the monotonicity of the generic $\mathbb{C}^*$-zero count, since the volume of the adjacency polytope is strictly increasing as new edges and nodes are added.

Corollary 3.8. *Let $(G_1, K_1, \vec{w}_1)$ and $(G_2, K_2, \vec{w}_2)$ be two connected networks such that $G_1 < G_2$. Then the generic $\mathbb{C}^*$-zero count of $\vec{f}_{G_1}$ is strictly less than that of $\vec{f}_{G_2}$.*

In addition, we also get that the phase-delayed algebraic Kuramoto system and lossy PV-type algebraic power flow system are Bernshtein-general.

Corollary 3.9. *For generic choices of $\overline{w}_i, a_{ij}, C_{ij}$, the algebraic Kuramoto systems with phase delays (2.7) is Bernshtein-general, and its $\mathbb{C}^*$-zero count equals to the normalized volume of ∇_G.*

Proof. As noted before, for generic choices of $\overline{w}_1, \ldots, \overline{w}_n$, all $\mathbb{C}^*$-zeros of this system are isolated and regular. Let η be the generic number of $\mathbb{C}^*$-zeros this system has. By Theorem 2.2 and Corollary 3.6, we know $\eta \leq \mathrm{Vol}(\check{\nabla}_G)$. By the Parameter Homotopy Theorem [30], the number of $\mathbb{C}^*$-zeros of this parametrized system constant on a nonempty Zariski open set of the parameters. By setting $C_{ij} = 1$, we recover (2.3) and by Theorem 3.5 and Corollary 3.6, this system is Bernshtein-general, with $\mathrm{Vol}(\check{\nabla}_G)$ $\mathbb{C}^*$-zeros. This shows that $\eta \geq \mathrm{Vol}(\check{\nabla}_G)$, giving the result. ∎

From the same argument, we get the generic zero count for the PV-type power flow system.

Corollary 3.10. *For generic choices of P_i, b_{ij}, g_{ij}, the lossy PV-type algebraic power flow system (2.6) is Bernshtein-general, and its $\mathbb{C}^*$-zero count equals to the normalized volume of $\check{\nabla}_G$.*

4. Explicit genericity conditions on coupling coefficients and refined zero count. So far we have established that the generic $\mathbb{C}^*$-zero count of the algebraic Kuramoto system $\vec{f}_G$ is the adjacency polytope bound $\mathrm{Vol}(\check{\nabla}_G)$. That is, for almost all choices of the parameters $\vec{w}$ (natural frequencies) and K (coupling coefficients), the $\mathbb{C}^*$-zero count for $\vec{f}_G$ equals $\mathrm{Vol}(\check{\nabla}_G)$.

This section aims to understand when the $\mathbb{C}^*$-zero count drops below the generic root count and by how much. We focus on the effect of coupling coefficients $\{k_{ij}\}$ while leaving the choices of the natural frequencies $\vec{w}$ generic, which ensures all $\mathbb{C}^*$-zeros of $\vec{f}_G$ and $\vec{f}^*_G$ are isolated and regular.

Definition 4.1. *Given a nontrivial and connected graph G with $n+1$ nodes, we define its* ***exceptional coupling coefficients*** *$\mathcal{K}(G)$ to be the set of symmetric and nonzero coupling coefficients $K = \{k_{ij}\}$ such that the number of isolated $\mathbb{C}^*$-zeros of $\vec{f}_{(G,K,\vec{w})} = 0$ is strictly less than $\mathrm{Vol}(\check{\nabla}_G)$, for any choice of $\vec{w} \in \mathbb{C}^n$.*

We first give an algebraic and graph-theoretic description for $\mathcal{K}(G)$. Then, for certain families of graphs, we decompose $\mathcal{K}(G)$ into strata, on which we develop refined root counts.

4.1. Exceptional coupling coefficients for a facial systems. We start with an analysis of a single facial system. For a given face F of $\check{\nabla}_G$, we describe the coupling coefficients for which the facial system $\mathrm{init}_F(\vec{f}^*_G)$ has a $\mathbb{C}^*$-zero, signaling the genericity condition for Theorem 3.5 is broken.

Definition 4.2. *Let $F \neq \varnothing$ be a proper face of $\check{\nabla}_G$ such that G_F is connected. We define the set*

$$\mathcal{K}(\vec{G}_F) = \{\boldsymbol{k}(\vec{G}_F) \in (\mathbb{C}^*)^{|F|} \mid \mathrm{init}_F(\vec{f}^*_G) = \vec{0} \text{ has a } \mathbb{C}^*\text{-solution}\}$$

to be the set of ***exceptional coupling coefficients*** *with respect to the face F (or $\vec{G}_F$).*

Recall that $\boldsymbol{k}(\vec{G}_F)$ is the vector whose entries are the coupling coefficients k_{ij} for arcs (i,j) in $\vec{G}_F$. By Bernshtein's Second Theorem (Theorem 2.2), if, for a proper face F of $\check{\nabla}_G$, the coupling coefficients $\boldsymbol{k}(\vec{G}_F) \in \mathcal{K}(\vec{G}_F)$, then the algebraic Kuramoto system $\vec{f}_G$ is not Bernshtein-general, and its isolated $\mathbb{C}^*$-zero count is strictly less than the adjacency polytope bound $\mathrm{Vol}(\check{\nabla}_G)$. Note that since we fixed a single face, this is inherently a local description of a patch of $\mathcal{K}(G)$ (Definition 4.1). By Theorems 2.2 and 3.5, $\mathcal{K}(\vec{G}_F)$ is contained in a proper and Zariski closed subset of Lebesgue measure zero[8]. In the following, we describe the structure of $\mathcal{K}(\vec{G}_F)$.

To lay down the algebraic foundation, we first show that every facial system of $\vec{f}^*_G$ can be transformed into a "cycle form", which can be understood more easily using graph theoretic information.

Lemma 4.3. *Let $F \neq \varnothing$ be a proper face of $\check{\nabla}_G$, $\check{Q}(\vec{G}_F)$ and $\boldsymbol{a}(\vec{G}_F)$ be the corresponding reduced incidence matrix of $\vec{G}_F$ and its coupling vector, respectively. Then $\mathrm{init}_F(\vec{f}^*_G)(\boldsymbol{x}) = \vec{0}$ if and only if*

$$\check{Q}(\vec{G}_F)\left(\boldsymbol{x}^{\check{Q}(\vec{G}_F)} \circ \boldsymbol{a}(\vec{G}_F)\right)^\top = \vec{0}.$$

This form will be referred to as the "cycle form" of the facial system $\mathrm{init}_F(\vec{f}^*_G)$ since the null space of the incidence matrix of a digraph is spanned by fundamental cycle vectors.

Proof. Recall that $\vec{f}^*_G = R \cdot \vec{f}_G$ for a nonsingular matrix $R = [r_{ij}]$ (see Subsection 2.6). Since we assumed $a_{ij} = a_{ji}$, for each $k = 1, \ldots, n$, we have

$$\mathrm{init}_F(f^*_{G,k})(\boldsymbol{x}) = \sum_{(i,j)\in\vec{G}_F} (r_{ki} - r_{kj})\, a_{ij}\, \boldsymbol{x}^{\vec{e}_i - \vec{e}_j} = \sum_{(i,j)\in\vec{G}_F} \langle\, \boldsymbol{r}_k\,,\, \vec{e}_i - \vec{e}_j \,\rangle\, a_{ij}\, \boldsymbol{x}^{\vec{e}_i - \vec{e}_j},$$

where $\boldsymbol{r}_k$ is the k-th row of the matrix R. Since vectors of the form $\vec{e}_i - \vec{e}_j$ for $(i,j) \in \mathcal{E}(\vec{G}_F)$ are exactly the columns in $\check{Q}(\vec{G}_F)$,

$$\mathrm{init}_F(\vec{f}^*_G)(\boldsymbol{x}) = R\,\check{Q}(\vec{G}_F)\,\left(\boldsymbol{x}^{\check{Q}(\vec{G}_F)} \circ \boldsymbol{a}(\vec{G}_F)\right)^\top.$$

Since the square matrix R is assumed to be nonsingular, this establishes the equivalence. ∎

[8] This result is justified more directly in an unlabelled paragraph in Bernshtein's paper [3].

This cycle form suggests a strong tie between the topological features of face subgraphs of G and the algebraic features of the initial systems of the randomized Kuramoto system $\vec{f}^*_G$. The rest of this paper is devoted to exploring this connection by leveraging the wealth of existing knowledge about the facial complex of $\check{\nabla}_G$. In particular, we will show that the cycle form of facial systems produces concrete genericity conditions on the coupling coefficients.

Note that a facial system $\operatorname{init}_F(\vec{f}^*_G)$ can always be decomposed into decoupled collections of facial systems corresponding to weakly connected components of $\vec{G}_F$. We state this as a lemma.

Lemma 4.4. *For a nonempty and proper face F of $\check{\nabla}_G$ whose facial subdigraph $\vec{G}_F$ consists of weakly connected components $\vec{H}_1, \dots, \vec{H}_\ell$, there exists nonempty faces $F_1, \dots, F_\ell$ of F (which are also faces of $\check{\nabla}_G$) such that $\vec{H}_i = \vec{G}_{F_i}$ for $i = 1, \dots, \ell$ and*

$$\operatorname{init}_F(\vec{f}^*_G)(\boldsymbol{x}) = \vec{0} \quad \Longleftrightarrow \quad \operatorname{init}_{F_i}(\vec{f}^*_G)(\boldsymbol{x}) = \vec{0} \quad \text{for each } i = 1, \dots, \ell.$$

Proof. Let $V_i = \mathcal{V}(\vec{H}_i)$ for $i = 1, \dots, \ell$ with $0 \in V_1$. We arrange the coordinates of $\mathbb{Z}^n$ according to the grouping $V_1, \dots, V_\ell$ so that an inner normal vector $\boldsymbol{\alpha} \in \mathbb{R}^n$ that defines the face F of $\check{\nabla}_G$ is written as $\boldsymbol{\alpha} = \begin{bmatrix} \boldsymbol{\alpha}_1 & \cdots & \boldsymbol{\alpha}_\ell \end{bmatrix}$ where $\boldsymbol{\alpha}_i \in \mathbb{R}^{|V_i|}$ corresponds to nodes in V_i. Then for $i = 1, \dots, \ell$,

$$\boldsymbol{v}_i := \begin{bmatrix} \mathbf{0}_{|V_1|+\cdots+|V_{i-1}|} & \boldsymbol{\alpha}_i & \mathbf{0}_{|V_{i+1}|+\cdots+|V_\ell|} \end{bmatrix} \in \mathbb{R}^n$$

defines a face F_i of $\check{\nabla}_G$ such that $\vec{H}_i = \vec{G}_{F_i}$.

By grouping the rows and columns of $\check{Q}(\vec{G}_F)$ according to the nodes and arcs in $H_1, \dots, H_\ell$, $\check{Q}(\vec{G}_F)$ has the block structure

$$\check{Q}(\vec{G}_F) = \begin{bmatrix} \check{Q}(\vec{H}_1) & & \\ & \ddots & \\ & & \check{Q}(\vec{H}_\ell) \end{bmatrix}.$$

Therefore, the cycle form of $\operatorname{init}_F(\vec{f}^*_G) = \vec{0}$, i.e., $\check{Q}(\vec{G}_F) \left(\boldsymbol{x}^{\check{Q}(\vec{G}_F)} \circ \boldsymbol{a}(\vec{G}_F) \right)^\top = \vec{0}$, is equivalent to

$$\check{Q}(\vec{H}_i) \left(\boldsymbol{x}_i^{\check{Q}(\vec{H}_i)} \circ \boldsymbol{a}(\vec{H}_i) \right)^\top = \vec{0} \quad \text{for each } i = 1, \dots, \ell,$$

where $\boldsymbol{x}_i$ contain the coordinates corresponding to nodes in $\vec{H}_i$. This is equivalent to $\operatorname{init}_{F_i}(\vec{f}^*_G)(\boldsymbol{x}) = \vec{0}$ for each $i = 1, \dots, \ell$, by Lemma 4.3. ∎

Based on this observation, it is sufficient to consider facial systems corresponding to connected facial subgraphs. We now give a description of the coupling coefficients for which the facial system $\operatorname{init}_F(\vec{f}^*_G)$ has a $\mathbb{C}^*$-zero. Recall, a *bridge* of a graph is an edge that is not contained in any cycle of the graph. A graph that contains no bridges is said to be *bridgeless*. With respect to a spanning tree T of a connected graph, each edge outside of T induces a unique cycle in this graph containing this edge. We call the collection of such cycles the *fundamental cycles* with respect to T.

Theorem 4.5. *Consider a corank-d face F of $\check{\nabla}_G$ such that G_F is connected.*

- *If G_F contains a bridge, then $\mathcal{K}(\vec{G}_F) = \varnothing$.*
- *If G_F is bridgeless, for a choice of a spanning tree T of G_F, let $C_1, \dots, C_d < G_F$ be the fundamental cycles with respect to T and let $\vec{\eta}_1, \dots, \vec{\eta}_d$ be their associated primitive cycle vectors with respect to $\vec{G}_F$. We define the very affine variety $V(\vec{G}_F)$ to be the set of $(\boldsymbol{k}, \boldsymbol{v}) \in (\mathbb{C}^*)^{2|F|}$ satisfying*

$$\begin{cases} \boldsymbol{k}^{\vec{\eta}_i} = \boldsymbol{v}^{\vec{\eta}_i} & \text{for } i = 1, \dots, d. \\ \boldsymbol{v}\, \check{Q}(\vec{G}_F)^\top = \mathbf{0}. \end{cases}$$

Let $\pi : (\mathbb{C}^*)^{2|F|} \to (\mathbb{C}^*)^{|F|}$ be the projection onto the first $|F|$ coordinates (i.e. the $\boldsymbol{k}$ coordinates). Then $\mathcal{K}(\vec{G}_F) = \pi(V(\vec{G}_F))$.

Proof. First, we reduce the general statement to the special case in which G_F is spanning. After relabeling the nodes, we can assume $\mathcal{V}(G_F) = \{0, \dots, n'\}$, and $F \subset \mathbb{R}^{n'} \times \{\vec{0}\}$. If G_F is not spanning, i.e., $n > n'$, then

$$\check{Q}(\vec{G}_F) = \begin{bmatrix} \check{Q}(\vec{G}_{F'}) \\ \mathbf{0}_{(n-n')\times|F|} \end{bmatrix}.$$

Thus the cycle form of the facial system $\mathrm{init}_F(\vec{f}^*_G)(\boldsymbol{x}) = \vec{0}$, namely,

$$\vec{0} = \check{Q}(\vec{G}_F)\left(\boldsymbol{x}^{\check{Q}(\vec{G}_F)} \circ \boldsymbol{a}(\vec{G}_F)\right)^\top = \begin{bmatrix} \check{Q}(\vec{G}_{F'}) \\ \mathbf{0}_{(n-n')\times|F|} \end{bmatrix} \left(\boldsymbol{x}^{\begin{bmatrix} \check{Q}(\vec{G}_{F'}) \\ \mathbf{0}_{(n-n')\times|F|} \end{bmatrix}} \circ \boldsymbol{a}(\vec{G}_F)\right)^\top$$

is equivalent to

$$\vec{0} = \check{Q}(\vec{G}_{F'})\left((\boldsymbol{x}')^{\check{Q}(\vec{G}_{F'})} \circ \boldsymbol{a}(\vec{G}_{F'})\right)^\top \tag{4.1}$$

where $\boldsymbol{x}' = (x_1, \dots, x_{n'})$ with $(x_{n'+1}, \dots, x_n) \in (\mathbb{C}^*)^{n-n'}$ unconstrained. This equation is, in turn, equivalent to the facet system equation $\mathrm{init}_{F'}(\vec{f}^*_{G'})(\boldsymbol{x}') = \vec{0}$ by Lemma 4.3. Therefore, it is sufficient to assume that $n = n'$, i.e., G_F is spanning.

We now consider the first case: when G_F contains a bridge. Since the null space of $\check{Q}(\vec{G}_F)$ is spanned by primitive cycle vectors, every null vector must contain a zero coordinate, i.e., $\ker(\vec{G}_F) \cap (\mathbb{C}^*)^{|F|} = \varnothing$. Consequently, $\mathrm{init}_F(\vec{f}^*_G)(\boldsymbol{x}) = \vec{0}$, which is equivalent to

$$\check{Q}(\vec{G}_F)\left(\boldsymbol{x}^{\check{Q}(\vec{G}_F)} \circ \boldsymbol{a}(\vec{G}_{F'})\right)^\top = \vec{0}$$

has no $\mathbb{C}^*$-solutions for any choices $\boldsymbol{a}(\vec{G}_F) \in (\mathbb{C}^*)^{|F|}$. This establishes the first case.

For the second case, $G_F \le G$ spans G. Therefore, a spanning tree T of G_F is also a spanning tree of G. Let $\vec{T}$ be the digraph corresponding to T such that $\vec{T} <= \vec{G}_F$, and by rearranging the columns, we can write

$$\check{Q}(\vec{G}_F) = \begin{bmatrix} \check{Q}(\vec{T}) & \vec{v}_1 & \cdots & \vec{v}_d \end{bmatrix}$$
$$\boldsymbol{a}(\vec{G}_F) = \begin{bmatrix} \boldsymbol{a}(\vec{T}) & \alpha_1 & \cdots & \alpha_d \end{bmatrix}$$

where $\vec{v}_1, \dots, \vec{v}_d$ are the incidence vectors associated with arcs of $\vec{G}_F$ that are outside $\vec{T}$, and $\alpha_1, \dots, \alpha_d$ are their complex coupling coefficients. Let $\vec{\eta}_1, \dots, \vec{\eta}_d \in \mathbb{Z}^{|F|}$ be the primitive cycle vectors corresponding to the fundamental cycles induced by these arcs with respect to $\vec{H}$. Since they are only determined up to a choice of sign, we can assume

$$\vec{\eta}_i = \begin{bmatrix} \vec{\tau}_i \\ -\vec{e}_i \end{bmatrix} \quad \text{for some } \vec{\tau}_i \in \{-1, 0, +1\}^n,$$

in which $\vec{e}_i$ is considered as a vector in $\mathbb{Z}^d$. In other words, the first n coordinates (i.e., $\vec{\tau}_i$) of $\vec{\eta}_i$ are chosen so that $\vec{v}_i = \check{Q}(\vec{T})\,\tau_i$. By Lemma 2.4, $\langle\, \mathbf{1}\,,\, \vec{\eta}_i \,\rangle = 0$, therefore, $\langle\, \mathbf{1}\,,\, \vec{\tau}_i \,\rangle = 1$.

Suppose $\boldsymbol{k}(\vec{G}_F) \in \pi(V(\vec{G}_F))$, as defined above, we shall show $\mathrm{init}_F(\vec{f}^*_G)$ has a $\mathbb{C}^*$-zero. By assumption, there exists an $\boldsymbol{\eta} = \begin{bmatrix} \boldsymbol{\tau} & u_1 & \cdots & u_d \end{bmatrix} \in (\mathbb{C}^*)^{n+d}$ such that $\boldsymbol{\eta}^\top \in \ker \check{Q}(\vec{G}_F)$ and

$$\boldsymbol{k}(\vec{G}_F)^{\vec{\eta}_i} = (2\mathrm{i} \cdot \boldsymbol{a}(\vec{G}_F))^{\vec{\eta}_i} = (2\mathrm{i})^{\langle\, \mathbf{1}\,,\, \vec{\eta}_i \,\rangle} \circ \boldsymbol{a}(\vec{G}_F)^{\vec{\eta}_i} = \boldsymbol{a}(\vec{G}_F)^{\vec{\eta}_i} = \boldsymbol{\eta}^{\vec{\eta}_i} \quad \text{for } i = 1, \dots, d.$$

Note that with this partition of entries, $\boldsymbol{\eta}^{\vec{\eta}_i} = \boldsymbol{\tau}^{\vec{\tau}_i} \cdot u_i^{-1}$.

For any $\lambda \in \mathbb{C}^*$, define

$$\boldsymbol{x} = (\lambda \cdot \boldsymbol{\tau} \circ \boldsymbol{a}(\vec{T})^{-I})^{\check{Q}(\vec{T})^{-1}} \in (\mathbb{C}^*)^n.$$

Then from a straightforward calculation, we can verify that

$$\boldsymbol{x}^{\vec{v}_i} = \boldsymbol{x}^{\check{Q}(\vec{T})\vec{\tau}_i} = (\lambda \cdot \boldsymbol{\tau} \circ \boldsymbol{a}(\vec{T})^{-I})^{\vec{\tau}_i}. = \lambda \cdot \boldsymbol{\tau}^{\vec{\tau}_i} \circ \boldsymbol{a}(\vec{T})^{-\vec{\tau}_i}. = \lambda\, u_i \cdot \boldsymbol{\eta}^{\vec{\eta}_i} \circ \boldsymbol{a}(\vec{T})^{-\vec{\tau}_i},$$

and thus

$$\boldsymbol{x}^{\check{Q}(\vec{G}_F)} \circ \boldsymbol{a}(\vec{G}_F) = \begin{bmatrix} \boldsymbol{x}^{\check{Q}(\vec{T})} \circ \boldsymbol{a}(\vec{T}) & \boldsymbol{x}^{\vec{v}_1} \cdot \alpha_1 & \cdots & \boldsymbol{x}^{\vec{v}_d} \cdot \alpha_d \end{bmatrix} = \lambda \cdot \begin{bmatrix} \boldsymbol{\tau} & u_1 & \cdots & u_d \end{bmatrix} = \lambda\, \boldsymbol{\eta}.$$

That is, $\check{Q}(\vec{G}_F)(\boldsymbol{x}^{\check{Q}(\vec{G}_F)} \circ \boldsymbol{a}(\vec{G}_F))^\top = \vec{0}$. By Lemma 4.3, the facial system $\operatorname{init}_F \vec{f}^*_G$ has a $\mathbb{C}^*$-zero.

Conversely, suppose $\operatorname{init}_F(\vec{f}^*_G)$ has a zero $\boldsymbol{x} \in (\mathbb{C}^*)^n$. By Lemma 4.3,

$$\boldsymbol{\eta} := (\boldsymbol{x}^{\check{Q}(\vec{G}_F)} \circ \boldsymbol{a}(\vec{G}_F))^\top \in \ker \check{Q}(\vec{G}_F) \cap (\mathbb{C}^*)^{|F|}.$$

Since $\ker \check{Q}(\vec{G}_F)$ is spanned by $\vec{\eta}_1, \dots, \vec{\eta}_d$, $\boldsymbol{\eta} = \sum_{j=1}^d \lambda_j \vec{\eta}_j^\top$ for some $\lambda_1, \dots, \lambda_d \in \mathbb{C}$. This means

$$\boldsymbol{\eta}^{\vec{\eta}_i} = (\boldsymbol{x}^{\check{Q}(\vec{G}_F)} \circ \boldsymbol{a}(\vec{G}_F))^{\vec{\eta}_i} = \boldsymbol{x}^{\check{Q}(\vec{G}_F)\vec{\eta}_i} \circ \boldsymbol{a}(\vec{G}_F)^{\vec{\eta}_i} = \boldsymbol{x}^{\vec{0}} \circ \boldsymbol{a}(\vec{G}_F)^{\vec{\eta}_i} = \boldsymbol{a}(\vec{G}_F)^{\vec{\eta}_i}$$

for $i = 1, \dots, d$, since $\vec{\eta}_i \in \ker \check{Q}(\vec{G}_F)$. Also recall that $\boldsymbol{a}(\vec{G}_F) = \frac{1}{2\mathfrak{i}}\boldsymbol{k}(\vec{G}_F)$ and $\langle\, \mathbf{1}\,,\, \vec{\eta}_i \,\rangle = 0$ for all $i = 1, \dots, d$, so $\boldsymbol{a}(\vec{G}_F)^{\vec{\eta}_i} = \boldsymbol{k}(\vec{G}_F)^{\vec{\eta}_i}$ giving the result. ∎

Remark 4.6. This proof provides an explicit construction of an orbit in the $\mathbb{C}^*$-zero set of the facial system $\operatorname{init}_F(\vec{f}^*_G)$ from which Section 5 will derive interesting constructions.

Remark 4.7. Since, in the context of Theorem 4.5, the null space of $\check{Q}(\vec{G}_F)$ is spanned by the primitive cycle vectors $\vec{\eta}_1, \dots, \vec{\eta}_d$, the set $V(\vec{G}_F)$ can also be more conveniently be defined by

$$\boldsymbol{k}^{\vec{\eta}_i} = \left(\sum_{j=1}^d \lambda_j \vec{\eta}_j^\top\right)^{\vec{\eta}_i} \quad \text{for } i = 1, \dots, d \qquad \text{and} \qquad \sum_{j=1}^d \lambda_j \vec{\eta}_j^\top \in (\mathbb{C}^*)^{|F|}$$

with $\lambda_1, \dots, \lambda_d \in \mathbb{C}$ being part of the coordinates. We will use the two versions interchangeably.

The complexity of the description of $\mathcal{K}(G_F)$ amounts to the complexity of computing the projection $\pi(V(\vec{G}_F))$. We now describe cases where this projection can be computed explicitly.

A corank-0 face F of $\check{\nabla}_G$ is a simplicial face, i.e., an affinely independent face. Its corresponding facial subgraph G_F must be a forest and hence not bridgeless.

Corollary 4.8. *For a proper and nonempty corank-0 face F of $\check{\nabla}_G$, $\mathcal{K}(\vec{G}_F) = \varnothing$, and the corresponding facial system $\operatorname{init}_F(\vec{f}^*_G)$ has no $\mathbb{C}^*$-zero for any choice of (nonzero) coupling coefficients.*

We now consider the case of a facial subgraph consisting of d cycles sharing a single edge.

Proposition 4.9. *For an integer $d \geq 1$, let F be a proper and nonempty corank-d face of $\check{\nabla}_G$ for which G_F consists of of d cycles $\{C_1, \dots, C_d\}$ that share exactly one edge e (i.e., $\mathcal{E}(C_i) \cap \mathcal{E}(C_j) = \{e\}$ for any pair of distinct i, j), and let $\vec{\eta}_1, \dots, \vec{\eta}_d \in \{-1, 0, 1\}^{|F|}$ be the primitive cycle vectors of $\vec{G}_F$ corresponding to $C_1, \dots, C_d$ with consistent signs. Then $\operatorname{init}_F(\vec{f}^*)$ has a $\mathbb{C}^*$-zero if and only if*

$$(-1)^{|C_1|/2}\boldsymbol{k}(\vec{G}_F)^{\vec{\eta}_1} + \cdots + (-1)^{|C_d|/2}\boldsymbol{k}(\vec{G}_F)^{\vec{\eta}_d} = 1. \tag{4.2}$$

Here, the "consistent" sign assignment means the common nonzero entries in $\vec{\eta}_1, \dots, \vec{\eta}_d$ all share the same sign for $d > 1$. If $d = 1$, the set $\{\vec{\eta}_1\}$ that contains the only choice of primitive cycle vector, regardless of the signs, will still be considered as consistent for simplicity.

Proof. Recall that the face subgraph G_F must be bipartite (Theorem 2.3) and therefore the cycles $C_1, \dots, C_d$ must be even. For $i = 1, \dots, d$, let $m_i = |C_i|$, $T_i = C_i - \{e\}$, and $\vec{T}_i < \vec{G}_F$ be the corresponding subdigraph. We order the columns in the reduced incidence matrix $\check{Q}(\vec{G}_F)$ so that

$$\check{Q}(\vec{G}_F) = \begin{bmatrix} \check{Q}(\vec{T}_1) & \cdots & \check{Q}(\vec{T}_d) & \vec{v}_e \end{bmatrix},$$

where $\vec{v}_e$ is the incidence vector associated with the edge e in $\vec{G}_F$. Since e is contained $C_1, \dots, C_d$, there is a unique vector $\vec{\tau}_i$, for each $i = 1, \dots, d$, such that $\vec{v}_e = \check{Q}(\vec{T}_i)\,\vec{\tau}_i$. Since $\operatorname{corank}(F) = d$, $\ker \check{Q}(\vec{G}_F)$ is spanned by the d primitive cycle vectors (expressed as row vectors)

$$\boldsymbol{\eta}_i = \Big[\overbrace{0 \ \cdots \ 0}^{\sum_{j=1}^{i-1} m_j} \ \vec{\tau}_i^{\top} \ \overbrace{0 \ \cdots \ 0}^{\sum_{j=i+1}^{d} m_j} \ -1 \Big] \qquad \text{for } i = 1, \dots, d$$

corresponding to the cycles $C_1, \dots, C_d$. Recall that, by Lemma 2.4, $\langle \mathbf{1}, \vec{\eta}_i \rangle = 0$, and thus there must be $m_i/2$ positive entries and $m_i/2 - 1$ negative entries in $\vec{\tau}_i$. That is, $(\vec{\tau}_i^{\top})^{\vec{\tau}_i} = (-1)^{m_i/2-1}$.

Then any $\boldsymbol{v}^{\top} \in \ker \check{Q}(\vec{G}_F) \cap (\mathbb{C}^*)^{|F|}$ can be expressed as

$$\boldsymbol{v} = \sum_{i=1}^{d} \lambda_i \boldsymbol{\eta}_i = \begin{bmatrix} \lambda_1 \vec{\tau}_1^{\top} & \cdots & \lambda_d \vec{\tau}_d^{\top} & -(\lambda_1 + \cdots + \lambda_d) \end{bmatrix}$$

for some $\lambda_i \in \mathbb{C}^*$ such that $\lambda_1 + \cdots + \lambda_d \neq 0$. In this case, it is easy to verify that

$$\boldsymbol{v}^{\vec{\eta}_i} = \frac{(-1)^{m_i/2} \lambda_i}{\lambda_1 + \cdots + \lambda_d}$$

By Remark 4.7, the exceptional coupling coefficients $\boldsymbol{k} := \boldsymbol{k}(\vec{G}_F)$ are defined by the equations

$$(-1)^{m_i/2}\, \boldsymbol{k}^{\vec{\eta}_i} = \frac{\lambda_i}{\lambda_1 + \ldots + \lambda_d} \qquad \text{for } i = 1, \dots, d, \tag{4.3}$$

and $\mathcal{K}(\vec{G}_F)$ is the projection of the solution set of this system onto the $\boldsymbol{k}$ coordinates. It suffices to show that this projection is exactly the hypersurface, $\mathcal{H}$, defined by (4.2).

Summing equations (4.3) for $i = 1, \dots, d$, we get the equation (4.2) and therefore, $\mathcal{K}(\vec{G}_F) \subseteq \mathcal{H}$. Conversely, for any $\boldsymbol{k} \in \mathcal{H}$, let $\lambda_i = (-1)^{m_i/2} \boldsymbol{k}^{\vec{\eta}_i} \in \mathbb{C}^*$ for $i = 1, \dots, d$. Then $\lambda_1 + \cdots + \lambda_d = 1 \neq 0$, and $(-1)^{m_i/2} \boldsymbol{k}^{\vec{\eta}_i} = \frac{\lambda_i}{\lambda_1 + \cdots + \lambda_d}$, satisfying (4.3) so $\boldsymbol{k} \in \mathcal{K}(\vec{G}_F)$. ∎

In the special case of $d = 1$ (i.e. G_F contains a unique cycle) the above proposition can be reduced to a particular simple binomial description for $\mathcal{K}(\vec{G}_F)$.

Corollary 4.10. *Let F be a corank-1 face of $\check{\nabla}_G$ for which G_F is connected, $\vec{\eta}$ be a primitive cycle vector of $\vec{G}_F$, and $\boldsymbol{k}(\vec{G}_F)$ be the corresponding coupling vector. The facial system $\operatorname{init}_F(\vec{f}_G^*)$ has a $\mathbb{C}^*$-zero if and only if G_F is a cycle and*

$$\boldsymbol{k}(\vec{G}_F)^{\vec{\eta}} = (-1)^{|F|/2} \tag{4.4}$$

4.2. Global description of exceptional coupling coefficients. The set $\mathcal{K}(\vec{G}_F)$ characterizes the coupling coefficients that will cause a specific facial system of $\vec{f}_G^*$ to have $\mathbb{C}^*$-zero. By Bernshtein's Second Theorem (Theorem 2.2), the $\mathbb{C}^*$-root count of $\vec{f}_G^*$ drops below the adjacency polytope bound if and only if the coupling coefficients are in $\mathcal{K}(\vec{G}_F)$ for some nonempty and proper face F of $\check{\nabla}_G$. By taking the union of all such faces, this gives a global description of the exceptional coupling coefficients $\mathcal{K}(G)$ for which the genericity condition in Theorem 3.5 is broken.

Note, however, that most faces have no contribution to the set of exceptional coupling coefficients. Combining Theorem 2.3 and Theorem 4.5, we get the following topological constraints on the faces with potentially nontrivial contribution to $\mathcal{K}(G)$.

Proposition 4.11. *Let Φ be the subset of nontrivial and proper faces of $\check{\nabla}_G$ such that their facial subgraphs are*

1. *bridgeless (and hence cyclic), and*
2. *maximally bipartite in their induced subgraphs.*

Then the set of exceptional coupling coefficients is the union

$$\mathcal{K}(G) = \bigcup_{F \in \Phi} \mathcal{K}(\vec{G}_F).$$

In this union, subsets $\mathcal{K}(\vec{G}_F)$ are assumed to be embedded in the common ambient space that contains all coupling coefficients. Based on this and the local description from Proposition 4.9, we can derive the global description of the set of exceptional coupling coefficients for graphs consisting of multiple cycles sharing a single edge. The geometry of the adjacency polytope for such networks was studied in detail by D'Ali, Delucchi, and Michałek [17] and the closely related work [14].

As noted in the proof of Proposition 4.9, it is sufficient to focus on even cycles, since odd cycles have no contributions to the exceptional coupling coefficients.

Proposition 4.12. *Let G be a graph consisting of d independent even cycles, $C_1, \dots, C_d$ that overlap at a single edge e (i.e., $\mathcal{E}(C_i) \cap \mathcal{E}(C_j) = \{e\}$ for any $i \neq j$). Let $U_i \subset \{-1, 0, +1\}^{|\mathcal{E}(G)|}$, for $i = 1, \dots, d$ be the set of balanced primitive cycle vectors associated with $C_1, \dots, C_d$. Then the set of exceptional coupling coefficients $\mathcal{K}(G)$ is the union of subsets*

$$\left\{ \boldsymbol{k} \;:\; \sum_{j=1}^{d} (-1)^{|C_{i_j}|/2} \boldsymbol{k}^{\vec{\eta}_{i_j}} = 1 \text{ where } (\vec{\eta}_{i_1}, \dots, \vec{\eta}_{i_t}) \in U_{i_1} \times \cdots \times U_{i_t} \text{ is consistent} \right\}. \tag{4.5}$$

Here, *balanced* means each $\vec{\eta}_{i_j}$ satisfies $\langle \mathbf{1}, \vec{\eta}_{i_j} \rangle = 0$ (has equal numbers of -1 and $+1$). And the tuple $(\vec{\eta}_{i_1}, \dots, \vec{\eta}_{i_t})$ is considered *consistent* if $\{i_1, \dots, i_t\}$ are distinct and for each pair i_j and $i_{j'}$, the common nonzero entries of $\vec{\eta}_{i_j}$ and $\vec{\eta}_{i_{j'}}$ have the same signs, as in Proposition 4.9.

Proof. By Proposition 4.11, the only contributions to $\mathcal{K}(G)$ come from nonempty proper faces associated with face subgraphs that are bridgeless and maximally bipartite in their induced subgraphs. Let Φ be the set of such faces (as in Proposition 4.11). In this case, these are face subgraphs consisting of exactly a subset of the cycles $C_1, \dots, C_d$.

It suffices to show that consistent tuples $(\vec{\eta}_{i_1}, \dots, \vec{\eta}_{i_t}) \in U_{i_1} \times \cdots \times U_{i_t}$ produce the primitive cycle vectors for all facial subdigraphs $\{\vec{G}_F \mid F \in \Phi\}$, since, by Proposition 4.9, each subset is simply $\mathcal{K}(\vec{G}_F)$ in that case. This is given by [12, Theorem 7] which says that for each $(\vec{\eta}_{i_1}, \dots, \vec{\eta}_{i_t}) \in U_{i_1} \times \cdots \times U_{i_t}$ there exists a corank-t face F of $\check{\nabla}_G$ such that $\vec{\eta}_{i_1}, \dots, \vec{\eta}_{i_t}$ are the primitive cycle vectors of $\vec{G}_F$, and vice versa. With this, the statement is an implication of Proposition 4.9 and Proposition 4.11. ∎

4.3. Refined root count for unicycle networks. G is a unicycle graph if it contains a unique cycle. For such a graph, Corollary 4.10 can be sharpened into a condition that detects and quantifies the drop in the $\mathbb{C}^*$-root count of $\vec{f}_G$ relative to the adjacency polytope bound.

Intuitively speaking, all coupling coefficients in $\mathcal{K}(G)$ are exceptional, in the sense that they cause the $\mathbb{C}^*$-zero count of the algebraic Kuramoto system to drop below the adjacency polytope bound. However, some are "more exceptional" than others. In this section, we develop this idea rigorously. We begin by restating the definition of the exceptional coupling coefficients with respect to a face in terms of network properties under the unicycle assumption.

Definition 4.13 (Balanced subnetwork). *For a unicycle graph G that contains a unique cycle O, let $\vec{O}^+$ and $\vec{O}^-$ be digraphs corresponding to assigning clockwise and counterclockwise orientations to O, respectively. For an acyclic subdigraph $\vec{H} < \vec{G}$, we say $(\vec{H}, K)$ is a* ***balanced*** *directed acyclic subnetwork of (G, K) if*

- $|\mathcal{E}(\vec{H}) \cap \mathcal{E}(\vec{O}^+)| = |\mathcal{E}(\vec{H}) \cap \mathcal{E}(\vec{O}^-)| = |\mathcal{V}(O)|/2$*; and*
- *let $\kappa^+(\vec{H})$ and $\kappa^-(\vec{H})$ be the products of the coupling coefficients along the arcs in $\mathcal{E}(\vec{H}) \cap \mathcal{E}(\vec{O}^+)$ and $\mathcal{E}(\vec{H}) \cap \mathcal{E}(\vec{O}^-)$, respectively, then*

$$\frac{\kappa^+(\vec{H})}{\kappa^-(\vec{H})} = (-1)^{|\mathcal{V}(O)|/2}.$$

Since the only type of subnetworks of interest here are directed and acyclic, they will simply be referred to as "balanced subnetworks". Note that this definition implies that the unique cycle O must be an even cycle. For example, the subdigraph $\vec{H}$ of C_4 shown in Figure 4.1 is acyclic and contains two clockwise arcs k_{10}, k_{32} and two counterclockwise arcs k_{12}, k_{30}. Thus, $\kappa^+(\vec{H}) = k_{10}k_{32}$ and $\kappa^-(\vec{H}) = k_{12}k_{30}$, and it is considered a balanced subnetwork if $k_{10}k_{32} = k_{12}k_{30}$. Using this concept Corollary 4.10 can be restated into the following.

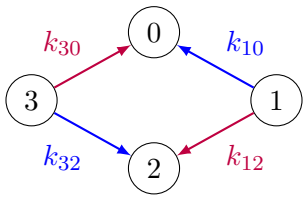


Figure 4.1: A balanced subnetwork of a 4-cycle

Proposition 4.14. *Let $(G, K, \vec{w})$ be a network in which G is a connected unicycle graph. For generic real or complex $\vec{w}$, the following are equivalent:*

1. *the $\mathbb{C}^*$-zero count of $\vec{f}_G$ is strictly less than $\mathrm{Vol}(\check{\nabla}_G)$;*
2. *$(G, K, \vec{w})$ has a balanced directed acyclic subnetwork.*

If the unique cycle in G is an odd cycle, then there can be no balanced subnetworks, and thus the $\mathbb{C}^*$-zero count of $\vec{f}_G$ must reach the adjacency polytope bound $\mathrm{Vol}(\check{\nabla}_G)$. Combined with the known results on adjacency polytope bound [13], we obtain the following stronger root count.

Corollary 4.15. *Let $(G, K, \vec{w})$ be a network in which G is a connected unicycle graph and its unique cycle O is an odd cycle. Then for any choices of real or complex $\vec{w}$ and $K = \{k_{ij} \in \mathbb{C}^* \mid \{i, j\} \in \mathcal{E}(G)\}$, the $\mathbb{C}^*$ zeroes of $\vec{f}_G$ are all isolated, and the total number, counting multiplicity, equals the adjacency polytope bound*

$$\mathrm{Vol}(\check{\nabla}_G) = 2^\ell \, m \binom{m-1}{(m-1)/2},$$

where $m = |\mathcal{V}(O)|$ and $\ell = |\mathcal{V}(G) \setminus \mathcal{V}(O)|$.

If the unique cycle O in G is even, then there may be balanced subnetworks. The usefulness of the concept of balanced subnetworks lies in the fact that each maximal balanced subnetwork reduces the $\mathbb{C}^*$-zero count of $\vec{f}_G$ by one. Here, maximal balanced subnetworks are simply maximal elements in the poset of balanced subnetworks. To establish this, we will utilize the facet decomposition homotopy [9, Theorem 3], which we summarize as the following lemma with minor modifications.

Lemma 4.16 (Adapted from Theorem 3 [9]). *For generic choices of real or complex $\vec{w}$ and symmetric, nonzero coupling coefficients $K = \{k_{ij} \mid \{i, j\} \in \mathcal{E}(G)\}$, let $\vec{f}_G$ be the algebraic Kuramoto system derived from $(G, K, \vec{w})$. Then the number of isolated $\mathbb{C}^*$-zeros that $\vec{f}_G$ has equals to the sum over all facets $F \in \mathcal{F}(\check{\nabla}_G)$ of the $\mathbb{C}^*$-zero count for pyramid systems of the form*

$$\check{Q}(\vec{G}_F) \left(\boldsymbol{x}^{\check{Q}(\vec{G}_F)} \circ \boldsymbol{a}(\vec{G}_F) \right)^\top = \vec{w}. \tag{4.6}$$

We call the above system a *pyramid system*[9] because the Newton polytope of this system is a pyramid of the form $\mathrm{conv}(\{\vec{0}\} \cup F)$.

Proposition 4.17. *Let $(G, K, \vec{w})$ be a network in which G is a connected unicycle graph, and its unique m-cycle O is an even cycle. Then for generic choices of real or complex $\vec{w}$, the $\mathbb{C}^*$-zeros of $\vec{f}_G$ are all isolated, and the total number equals*

$$2^\ell \left[m \binom{m-1}{m/2-1} - \beta \right],$$

where $\ell = |\mathcal{V}(G) \setminus \mathcal{V}(O)|$, and β is the number of maximally balanced subnetwork in $(G, K, \vec{w})$.

[9] In [9], the first named author named such a system a "facet system". This name, unfortunately, conflicts with the terms facial system and facet system we adopt here.

Proof. By Lemma 2.1 and an induction on leaf nodes, it is sufficient to assume $G = C_N$ and show the $\mathbb{C}^*$-zero count to be $N\binom{N-1}{N/2-1} - \beta$. Indeed, it was shown in [13, Theorem 13] that the adjacency polytope bound, i.e., the generic $\mathbb{C}^*$-zero count is $N\binom{N-1}{N/2-1}$. So we simply have to show each maximally balanced subnetwork reduces the $\mathbb{C}^*$-zero count by 1.

Fix a facet $F \in \mathcal{F}(\check{\nabla}_{C_N})$ that corresponds to $\vec{H} = (\vec{C}_N)_F$ in a maximally balanced subnetwork $(\vec{H}, K, \vec{w})$. That is, we assume

$$(-1)^{N/2} = \frac{\kappa^+(\vec{H})}{\kappa^-(\vec{H})} = \boldsymbol{a}(\vec{H})^{\vec{\eta}},$$

where $\vec{\eta}$ is a primitive cycle vector of $\vec{H}$. We shall first show that under this assumption the number of $\mathbb{C}^*$-solutions to the pyramid system (4.6) has is exactly $N/2-1$ (which is one less than the generic $\mathbb{C}^*$-zero count of the pyramid system $N/2$ as shown in [13, Proposition 12].)

We start by rewriting (4.6) in a convenient form. Fix an arbitrary arc e of $\vec{H}$, and let $\vec{T} = \vec{H} - \{e\}$, then the corresponding graph T is a spanning tree of C_N, and therefore $\check{Q}(\vec{T})$ is nonsingular. We shall arrange the entries so that $\check{Q}(\vec{H}) = \begin{bmatrix} \check{Q}(\vec{T}) & \vec{v}_e \end{bmatrix}$ and $\boldsymbol{a}(\vec{H}) = \begin{bmatrix} \boldsymbol{a}(\vec{T}) & a_e \end{bmatrix}$, and where $\vec{v}_e$ and a_e are the incidence vector and the complex coupling coefficients for the arc e, respectively. By choosing an orientation for the cycle, we can assume the last coordinate of the primitive cycle vector $\boldsymbol{\eta}$ is -1. That is, $\boldsymbol{\eta} = \begin{bmatrix} \boldsymbol{\eta}_{\vec{T}} & -1 \end{bmatrix}$, where $\boldsymbol{\eta}_{\vec{T}}$ is the incidence vector of the oriented cycle with respect to the arcs in $\vec{T}$. Let $\boldsymbol{y} = (\check{Q}(\vec{T})^{-1}\vec{w})^\top$, and $y_{n+1} = 0$, which can be assumed to be generic since $\vec{w}$ is generic. Then the pyramid system (4.6) is equivalent to

$$\boldsymbol{x}^{\check{Q}(\vec{T})} \circ \boldsymbol{a}(\vec{T}) = \boldsymbol{y} + \lambda \boldsymbol{\eta}_{\vec{T}} \tag{4.7}$$

$$\boldsymbol{x}^{\vec{v}_e} \cdot a_e = y_{n+1} - \lambda. \tag{4.8}$$

Recall that $\boldsymbol{a}(\vec{T}) \in (\mathbb{C}^*)^n$, so (4.7) is equivalent to

$$\boldsymbol{x}^{\check{Q}(\vec{T})} = \boldsymbol{a}(\vec{T})^{-I} \circ (\boldsymbol{y} + \lambda \boldsymbol{\eta}_{\vec{T}}).$$

Moreover, since $\check{Q}(\vec{T})$ is unimodular, it has a unique $\mathbb{C}^*$-solution for any given $\lambda \in \mathbb{C}$, as long as the right-hand-side is also in $(\mathbb{C}^*)^n$. That is, under this assumption, the value of $\boldsymbol{x}$, in a solution, is uniquely determined by the value of λ. Therefore, it is sufficient to solve for λ and count the distinct solutions.

Note that $\vec{v}_e = \check{Q}(\vec{T})\,\vec{\eta}_{\vec{T}}$, so substituting (4.7) into (4.8) produces

$$y_{n+1} - \lambda = \boldsymbol{x}^{\vec{v}_e} \cdot a_e = (\boldsymbol{x}^{\check{Q}(\vec{T})})^{\vec{\eta}_{\vec{T}}} \cdot a_e = (\boldsymbol{a}(\vec{T})^{-I} \circ (\boldsymbol{y} + \lambda \boldsymbol{\eta}_{\vec{T}}))^{\vec{\eta}_{\vec{T}}} \cdot a_e = \boldsymbol{a}(\vec{G}_F)^{-\vec{\eta}} \cdot (\boldsymbol{y} + \lambda \boldsymbol{\eta}_{\vec{T}})^{\vec{\eta}_{\vec{T}}}$$

The problem is now reduced to the problem of counting the number of $\lambda \in \mathbb{C}$ that satisfies this equation. Let $I^+ = \{i \in [n] \mid \eta_i = +1\}$ and $I^- = \{i \in [n] \mid \eta_i = -1\}$, respectively. Then, the above equation is equivalent to

$$\prod_{i \in I^-} (y_i - \lambda) = \boldsymbol{a}(\vec{G}_F)^{-\vec{\eta}} \prod_{i \in I^+} (y_i + \lambda),$$

which is a polynomial equation in λ of degree up to $N/2$, i.e., up to $N/2$ solutions for λ. However, we will show this upper bound is not attainable.

Note that the degree $N/2$ terms of the two sides of the above equation are $(-1)^{N/2}\lambda^{N/2}$ and $\boldsymbol{a}(\vec{G}_F)^{-\vec{\eta}}\lambda^{N/2}$, respectively. Since we assumed that $\boldsymbol{a}(\vec{G}_F)^{\vec{\eta}} = (-1)^{N/2}$, the degree $N/2$ terms cancel, and therefore the above equation is actually a polynomial equation in λ of degree up to $N/2 - 1$ with the coefficients being symmetric functions in $\boldsymbol{y}$, which are assumed to be generic. Consequently, there are exactly $N/2 - 1$ solutions to this equation in λ, which produces $N/2 - 1$

distinct solutions to the system (4.7)-(4.8). That is, the pyramid system induced by the facet F has $N/2-1$ distinct $\mathbb{C}^*$-zeros.

So far, we have shown that, for any facet associated with a maximally balanced subnetwork, the pyramid system has $N/2-1$ $\mathbb{C}^*$-solutions, whereas a pyramid system induced by other facets has $N/2$ $\mathbb{C}^*$-solutions. It was established in [13, Proof of Theorem 13] that $\tilde{\nabla}_{C_N}$ has $2\binom{N-1}{N/2-1}$ facets. Therefore, the sum of the number of $\mathbb{C}^*$-solutions to each pyramid system defined in (4.6) is:

$$\beta\left(\frac{N}{2}-1\right)+\left(2\binom{N-1}{N/2-1}-\beta\right)\frac{N}{2}=N\binom{N-1}{N/2-1}-\beta.$$

An application of Lemma 4.16 then gives the result. ∎

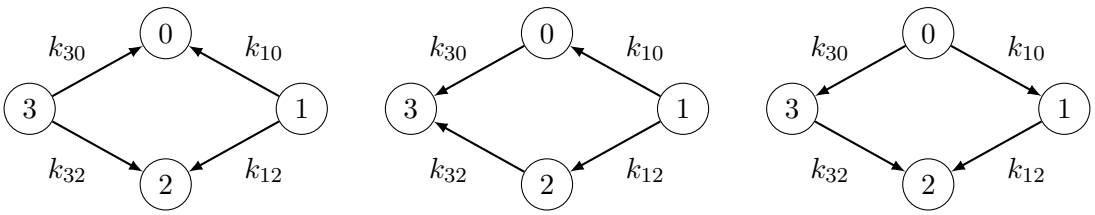


Figure 4.2: Representatives of the classes of balanced subnetworks in C_4

Example 4.18 (The 4-cycle case). Consider the case of $G=C_4$. As shown in [13, Theorem 16] and Theorem 3.5, for generic choices of $\vec{w}$ and K, $\vec{f}_G$ has 12 $\mathbb{C}^*$-zeros. This is the adjacency polytope bound. Now, with the choice of $\vec{w}$ remain generic, Proposition 4.17 provides a stratification of the space of coupling coefficients according to the maximum $\mathbb{C}^*$-zero count.

This network has up 6 balanced subnetworks supported by the digraphs shown in Figure 4.2 and their transposes. With these, we have the following decomposition of the coupling coefficients:

1. Consider the 1-dimensional "balancing variety" defined by the binomial system

$$\begin{cases} k_{10}k_{12}k_{32}^{-1}k_{30}^{-1}=1\\ k_{10}k_{12}^{-1}k_{32}^{-1}k_{30}=1\\ k_{10}k_{12}^{-1}k_{32}k_{30}^{-1}=1, \end{cases}$$

which will produce all 6 balanced subnetworks. It can be parametrized by

$$k_{10}=k_{12}=k_{32}=k_{30}=s,$$

for $s\in\mathbb{C}^*$. Any choice of $s\in\mathbb{C}^*$ produces an algebraic Kuramoto system $\vec{f}_G$ with at most $12-6=6$ $\mathbb{C}^*$-zeros. For generic s, the $\mathbb{C}^*$-zero count is exactly 6. Moreover, taking $s=1$ and $\omega_1=1.1,\omega_2=-2.1,\omega_3=1$, all six $\mathbb{C}^*$-zeros can be *real* (see Appendix B).

2. There are three 2-dimensional balancing varieties defined by two out of three binomial equations from the above system:

$$\begin{cases} k_{10}k_{12}k_{32}^{-1}k_{30}^{-1}=1\\ k_{10}k_{12}^{-1}k_{32}^{-1}k_{30}=1 \end{cases}\qquad \begin{cases} k_{10}k_{12}^{-1}k_{32}^{-1}k_{30}=1\\ k_{10}k_{12}^{-1}k_{32}k_{30}^{-1}=1, \end{cases}\qquad \begin{cases} k_{10}k_{12}k_{32}^{-1}k_{30}^{-1}=1\\ k_{10}k_{12}^{-1}k_{32}k_{30}^{-1}=1, \end{cases}$$

each producing 4 balanced subnetworks. They contain choices of coupling coefficients for which $\vec{f}_G$ has at most 8 $\mathbb{C}^*$-zeros. A generic choice of coupling coefficients on this variety will produce exactly 8 $\mathbb{C}^*$-zeros. The first of the three, can be parametrized by

$$k_{10}=s\qquad k_{12}=t\qquad k_{32}=\pm t\qquad k_{30}=\pm s,$$

where $(s,t)\in(\mathbb{C}^*)^2$. The other two, $\mathcal{K}(\vec{\eta}_2,\vec{\eta}_3)$ and $\mathcal{K}(\vec{\eta}_3,\vec{\eta}_1)$, can be parametrized similarly. Moreover, taking $s=1,t=-1.001,\omega_1=1.1,\omega_2=-2.1,\omega_3=1$ all 8 solutions can be real.

3. There are three 3-dimensional balancing varieties defined by binomial equations

$$k_{10}k_{12}k_{32}^{-1}k_{30}^{-1}=1 \qquad k_{10}k_{12}^{-1}k_{32}^{-1}k_{30}=1 \qquad k_{10}k_{12}^{-1}k_{32}k_{30}^{-1}=1,$$

each producing 2 balanced subnetworks. They contain coupling coefficients for which $\vec{f}_G$ has at most 10 $\mathbb{C}^*$-zeros. A generic choice of coupling coefficients on one of these varieties gives exactly 10 $\mathbb{C}^*$-zeros. The first of the three can be parametrized by

$$k_{10}=s \qquad k_{12}=u/s \qquad k_{32}=t \qquad k_{30}=u/t,$$

for $(s,t,u)\in(\mathbb{C}^*)^3$. The other two can be parametrized similarly. Moreover, taking $u=1.01, s=1, t=-1.001, \omega_1=1.1, \omega_2=-2.1, \omega_3=1$, all 10 solutions can be real.

4. The remaining choices of K, form a Zariski-dense subset in the space of all coupling coefficients for C_4, and define an algebraic Kuramoto system with 12 $\mathbb{C}^*$-zeros.

Under the assumption that $\vec{w}$ is generic, there are no other possibilities.

Example 4.19 (The 6-cycle case). Similarly, for the case of $G=O=C_6$, it is known that for generic choices of $\vec{w}$ and K, $\vec{f}_G$ has 60 $\mathbb{C}^*$-zeros. There are up to 20 balanced subnetworks that form 10 transpose pairs, and they can be described by arc orientation vectors $\vec{\eta}_1,\dots,\vec{\eta}_{10}$:

$$\begin{bmatrix}+1\\+1\\+1\\-1\\-1\\-1\end{bmatrix},\ \begin{bmatrix}+1\\+1\\-1\\+1\\-1\\-1\end{bmatrix},\ \begin{bmatrix}+1\\+1\\-1\\-1\\+1\\-1\end{bmatrix},\ \begin{bmatrix}+1\\+1\\-1\\-1\\-1\\+1\end{bmatrix},\ \begin{bmatrix}+1\\-1\\+1\\+1\\-1\\-1\end{bmatrix},\ \begin{bmatrix}+1\\-1\\+1\\-1\\+1\\-1\end{bmatrix},\ \begin{bmatrix}+1\\-1\\+1\\-1\\-1\\+1\end{bmatrix},\ \begin{bmatrix}+1\\-1\\-1\\+1\\+1\\-1\end{bmatrix},\ \begin{bmatrix}+1\\-1\\-1\\+1\\-1\\+1\end{bmatrix},\ \begin{bmatrix}+1\\-1\\-1\\-1\\+1\\+1\end{bmatrix},$$

where the ordering of the coordinates corresponds to the ordering $\boldsymbol{k}(O)=[k_{01},k_{12},k_{23},k_{34},k_{45},k_{50}]$. The exceptional coupling coefficients for which the $\mathbb{C}^*$-root count of $\vec{f}_G$ deviates from the generic root count are contained in the zero sets of binomial equations of the form

$$\boldsymbol{k}(O)^{\vec{\eta}_i}=-1,$$

for $i\in[10]$. This indicates that the exceptional coupling coefficients must include some negative values, which corresponds to repulsive couplings. The full description of the strata of exceptional coupling coefficients will be lengthy, so we only examine two special case.

Consider the family of coupling coefficients

$$\boldsymbol{k}(O)=[\pm s,\dots,\pm s],$$

with an odd number of negative choices, parametrized by $s\in\mathbb{C}^*$. Then $\boldsymbol{k}(O)^{\vec{\eta}_i}=-1$ is satisfied for $i=1,\dots,10$. By Proposition 4.17, for any $s\in\mathbb{C}^*$ and generic choices of $\vec{w}$, the $\mathbb{C}^*$-zero count of $\vec{f}_G$ is no more than $6\cdot\binom{6-1}{6/2-1}-2\cdot 10=40$. This bound is exact for generic choice of $s\in\mathbb{C}^*$.

Similarly, for the family

$$\boldsymbol{k}(O)=[\pm s,\pm s,\pm t,\dots,\pm t]$$

with an odd number of negative choices, parametrized by $(s,t)\in(\mathbb{C}^*)^2$, the conditions $\boldsymbol{k}(O)^{\vec{\eta}_i}=-1$ is satisfied only for $i=5,6,7,8,9,10$. Therefore, the $\mathbb{C}^*$-zero count of $\vec{f}_G$ is no more than $6\cdot\binom{6-1}{6/2-1}-2\cdot 6=48$ for any $(s,t)\in(\mathbb{C}^*)^2$ and generic $\vec{w}$. It is exact for generic choices of (s,t).

5. Positive-dimensional synchronization configurations for bipartite networks. We now turn our attention to positive-dimensional zero sets of the algebraic Kuramoto equations. They represent synchronization configurations that have at least one degree of freedom. We generalize existing constructions of positive-dimensional zero sets for the Kuramoto equations [1, 16, 28, 32] and characterize conditions under which they arise.

The crucial observation that enables our constructions is that when G is bipartite, by Theorem 2.3, G itself is a facet subgraph, i.e., there exists a $F\in\mathcal{F}(\check{\nabla}_G)$ such that $G=G_F$. Indeed,

$$\check{\nabla}_G=\{0\}\cup F\cup(-F).$$

That is, F and $-F$ contains the exponent vectors of all nonconstant terms. If we further assume the oscillators are homogeneous, i.e., all the natural frequencies are identical and thus the constant terms in $\vec{f}_G$ are all zero, then the algebraic Kuramoto system can be expressed in the simple form

$$\vec{f}_G(\boldsymbol{x}) = \check{Q}(G)(\boldsymbol{x}^{\check{Q}(\vec{G}_F)} \circ \boldsymbol{a}(G))^\top - \check{Q}(G)(\boldsymbol{x}^{-\check{Q}(\vec{G}_F)} \circ \boldsymbol{a}(G))^\top.$$

Note that both terms are cycle forms of facial systems, as shown in Lemma 4.3, and they correspond to $\operatorname{init}_F(\vec{f}^*_G)$ and $\operatorname{init}_{-F}(\vec{f}^*_G)$, respectively. In the previous section, we already constructed explicit solutions to $\operatorname{init}_F(\vec{f}^*_G)(\boldsymbol{x}) = \vec{0}$ under certain restrictions of the coupling coefficients. In the following, we investigate additional restrictions that will ensure the same solution also satisfies $\operatorname{init}_{-F}(\vec{f}^*_G)(\boldsymbol{x}) = \vec{0}$ and will therefore be a positive dimensional $\mathbb{C}^*$-zero set to $\vec{f}_G$ itself.

5.1. Unicycle graphs. We first consider the case when G is a unicycle graph. In this context, Proposition 4.14 can be paraphrased as the following necessary condition for the existence positive-dimensional $\mathbb{C}^*$-zero sets.

Proposition 5.1. *Let G be a (connected) unicycle graph that contains a unique cycle O. If the $\mathbb{C}^*$-zero sets of $\vec{f}_G$ is positive-dimensional, then O is even and (G, K) contains a balanced subnetwork.*

We now show that positive dimensional $\mathbb{C}^*$-zero sets always exist for unicycle networks of homogeneous oscillators that contain a balanced subnetwork.

Proposition 5.2 (Non-isolated $\mathbb{C}^*$-zero set for unicycle networks). *Let G be a (connected) unicycle graph on $n+1$ nodes that contains a unique even cycle O. Suppose the coupling coefficients satisfy:*

1. *(G, K) contains a balanced subnetwork $(\vec{H}, K)$;*
2. *$\boldsymbol{k}^2(O) = c^2 \cdot \mathbf{1}$ for some $c \in \mathbb{C}^*$, where $\boldsymbol{k}^2$ the element-wise square of $\boldsymbol{k}(O)$.*

Then for any $w \in \mathbb{C}$, the algebraic Kuramoto system $\vec{f}_G$ derived from homogeneous network $(G, K, w\,\vec{1})$ has a positive-dimensional $\mathbb{C}^*$-zero set. Moreover, after relabeling the nodes such that $\mathcal{V}(O) = \{0, 1, \ldots, m\}$, the $\mathbb{C}^*$-zero set of $\vec{f}_G$ contains one-dimensional components parametrized by

$$\begin{aligned}\begin{bmatrix} x_1(\lambda) & \cdots & x_m(\lambda)\end{bmatrix} &= (\lambda \cdot \boldsymbol{\eta}_{\vec{T}} \circ (\boldsymbol{a}(\vec{T})^{-I}))^{\check{Q}(\vec{T})^{-1}} \\ x_i &= \pm x_{\pi(i)} \quad \text{for } i = m+1, \ldots, n,\end{aligned}$$

where $\vec{T} < \vec{H}$ such that its corresponding graph T is a spanning tree of O, $[\boldsymbol{\eta}_{\vec{T}} \ -1]$ is a primitive cycle vector of $\vec{H}$, and $\pi(i)$ is the tree-order parent of node i.

Here, the choice $w\,\vec{1}$ for the coupling coefficients simply means that all natural frequencies $w_0, w_1, \ldots, w_n$ are identical, i.e., the network consists of homogeneous oscillators. Since the constant terms in the algebraic Kuramoto system (2.3) are their deviations $\overline{w}_i = w_i - \overline{w}$ from the mean, this corresponds to the requirement that all constant terms are zero. The tree-order parent node $\pi(i)$ of a node i outside the cycle O is its adjacent node in the unique path to the reference node 0.

Proof. By recursive applications of Lemma 2.1, $\mathbb{C}^*$-zeros for $\vec{f}_O$ always extend to $\mathbb{C}^*$-zeros for $\vec{f}_G$. So without loss, we can assume $G = O, n = m$. Let $F = \check{\nabla}_{\vec{H}}$, then F is a facet of $\check{\nabla}_G$, and,

$$\vec{f}_G(\boldsymbol{x}) = \check{Q}(\vec{H})(\boldsymbol{x}^{\check{Q}(\vec{H})} \circ \boldsymbol{a}(\vec{H}))^\top - \check{Q}(\vec{H})(\boldsymbol{x}^{-\check{Q}(\vec{H})} \circ \boldsymbol{a}(\vec{H}))^\top \tag{5.1}$$

Let T be a spanning tree of $O = G$ and let $\vec{T}$ be the corresponding subdigraph of $\vec{H}$. From the proof of Theorem 4.5, $\vec{T}$ gives rise to a cocharacter $\boldsymbol{\mu} = \mathbf{1}\check{Q}(\vec{T})^{-1}$. It provides the parametrization

$$\boldsymbol{x}(\lambda) = (\lambda \cdot \boldsymbol{\eta}_{\vec{T}} \circ \boldsymbol{a}(\vec{T})^{-I})^{\check{Q}(\vec{T})^{-1}} = (\boldsymbol{\eta}_{\vec{T}} \circ \boldsymbol{a}(\vec{T})^{-I})^{\check{Q}(\vec{T})^{-1}} \circ \lambda^{\boldsymbol{\mu}}$$

of a 1-dimensional orbit in the $\mathbb{C}^*$-zero set of $\operatorname{init}_F(\vec{f}^*_G)$ and therefore $\check{Q}(\vec{H})(\boldsymbol{x}(\lambda)^{\check{Q}(\vec{H})} \circ \boldsymbol{a}(\vec{H}))^\top = \vec{0}$. That is, the first term of (5.1) vanishes. With a straightforward calculation, we verify

$$\boldsymbol{x}(\lambda)^{-\check{Q}(\vec{H})} \circ \boldsymbol{a}(\vec{H}) = ((\lambda \cdot \boldsymbol{\eta}_{\vec{T}} \circ \boldsymbol{a}(\vec{T})^{-I})^{\check{Q}(\vec{T})^{-1}})^{\begin{bmatrix}-\check{Q}(\vec{T}) & -\check{Q}(\vec{T})\vec{\eta}_{\vec{T}}\end{bmatrix}} \circ \boldsymbol{a}(\vec{H}) = -\frac{c^2}{4\lambda} \cdot \boldsymbol{\eta},$$

where c is the constant such that $\boldsymbol{k}^2(\vec{H}) = c^2 \cdot \mathbf{1}$ from the assumption. Therefore,

$$\check{Q}(\vec{H})(\boldsymbol{x}(\lambda)^{-\check{Q}(\vec{H})} \circ \boldsymbol{a}(\vec{H}))^\top = \vec{0}.$$

Combined with the calculation above, we see that $\vec{f}_G(\boldsymbol{x}(\lambda)) = \vec{0}$ for all $\lambda \in \mathbb{C}^*$ and thus the $\mathbb{C}^*$-zero set of $\vec{f}_G = 0$ is positive-dimensional. ∎

The $\mathbb{C}^*$-orbit constructed here produces a positive-dimensional real zero set to the original transcendental Kuramoto system.

Proposition 5.3 (Non-isolated real zero set for unicycle networks). *Let G be a (connected) unicycle graph that contains a unique even cycle O. Suppose the choice of the coupling coefficients satisfies*

1. *(G, K) contains a balanced subnetwork $(\vec{H}, K)$;*
2. *$\boldsymbol{k}^2(O) = c^2 \cdot \mathbf{1}$ for some $c \in \mathbb{R}^+$.*

Then for any $w \in \mathbb{R}$, transcendental Kuramoto system (2.2) derived from homogeneous network $(G, K, w\vec{1})$ has positive-dimensional real zero sets.

Proof. Let $\vec{T}$ and $\boldsymbol{\eta}_{\vec{T}}$ be as defined in the above proof. Then for any $\lambda \in \mathbb{R}^*$,

$$\frac{ce^{\mathrm{i}\lambda}}{2\mathrm{i}} \cdot \boldsymbol{\eta}_{\vec{T}} \circ \boldsymbol{a}(\vec{T})^{-I} = \frac{ce^{\mathrm{i}\lambda}}{2\mathrm{i}} \cdot \boldsymbol{\eta}_{\vec{T}} \circ \left(\frac{1}{2\mathrm{i}}\boldsymbol{k}(\vec{T})\right)^{-I} \in (S^1)^m.$$

Since the group automorphism $\boldsymbol{x} \mapsto \boldsymbol{x}^{\check{Q}(\vec{T})^{-1}}$ preserves the the real torus $(S^1)^m \subset (\mathbb{C}^*)^m$, the restriction of the parametrized zero set $\boldsymbol{x}$, defined as

$$\boldsymbol{x}(\lambda) = \left(\frac{ce^{\mathrm{i}\lambda}}{2\mathrm{i}} \cdot \boldsymbol{\eta}_{\vec{T}} \circ \boldsymbol{a}(\vec{T})^{-I}\right)^{\check{Q}(\vec{T})^{-1}} \in (S^1)^m$$

for any $\lambda \in \mathbb{R}$. By Proposition 5.2 $\boldsymbol{x}(\lambda)$ also satisfies the algebraic Kuramoto equations. Therefore, $\boldsymbol{\theta}(\lambda) = \log(\boldsymbol{x}(\lambda))$ is a positive-dimensional real zero set of the transcendental Kuramoto system. ∎

Example 5.4 (4-cycle, again). For $G = C_4$, as noted in Example 4.18, there can be as many as 6 balanced subnetworks, depending on the choices of coupling coefficients. They come in 3 transpose-pairs whose representatives are shown in Figure 4.2. Each transpose-pair of balanced subnetworks produces an one-dimensional $\mathbb{C}^*$-zero sets of $\vec{f}_G = 0$ through the formula given in Proposition 5.2.

1. Consider the first subdigraph $\vec{H}_1$ in Figure 4.2. It is balanced if $k_{10}k_{32} = k_{12}k_{30}$. Then with the choice of $\vec{T}_1$ having arcs $(1,0), (1,2), (3,0)$, we can compute

$$\check{Q}(\vec{T}_1) = \begin{bmatrix} +1 & +1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & +1 \end{bmatrix} \qquad \check{Q}(\vec{T}_1)^{-1} = \begin{bmatrix} +1 & +1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & +1 \end{bmatrix} \qquad \boldsymbol{\eta}_{\vec{T}_1} = \begin{bmatrix} -1 & +1 & +1 \end{bmatrix},$$

   where the notation is as in Proposition 5.2. Then the function $\boldsymbol{x} : \mathbb{C}^* \to (\mathbb{C}^*)^3$, given by

$$\boldsymbol{x}(\lambda) = (\lambda \cdot \boldsymbol{\eta}_{\vec{T}_1} \circ \begin{bmatrix} a_{10} & a_{12} & a_{30} \end{bmatrix}^{-I})^{\check{Q}(\vec{T}_1)^{-1}} = \begin{bmatrix} -\frac{2\mathrm{i}\lambda}{k_{10}} & -\frac{k_{12}}{k_{10}} & \frac{2\mathrm{i}\lambda}{k_{30}} \end{bmatrix}$$

   has image inside the $\mathbb{C}^*$-zero set of $\vec{f}_{C_4}$. In other words, the function

$$\boldsymbol{x}(\lambda) = \begin{bmatrix} -2\mathrm{i}\lambda/k_{10} & -k_{12}/k_{10} & +2\mathrm{i}\lambda/k_{30} \end{bmatrix} \qquad \text{if} \quad k_{10}k_{32} = k_{12}k_{30}$$

   parametrizes the open part of an one-dimensional orbit in the $\mathbb{C}^*$-zero set of $\vec{f}_{C_4}$. In the special case when $k_{ij} = 1$, this one-dimensional orbit is *balanced* in the sense of [32, Definition 3.1] and [32] provides analysis of the stability and geometry of such orbits.

2. The second subnetwork $\vec{H}_2$ in Figure 4.2 is balanced if $k_{10}k_{30} = k_{12}k_{23}$. With $\vec{T}_2 < \vec{H}_2$ given by arcs $(1,0), (1,2), (0,3)$ and by following the construction above, we have

$$\boldsymbol{x}(\lambda) = \begin{bmatrix} +2\mathrm{i}\lambda/k_{10} & -k_{12}/k_{10} & +k_{03}/2\mathrm{i}\lambda \end{bmatrix} \qquad \text{if} \quad k_{10}k_{30} = k_{12}k_{23}$$

   whose image is in the $\mathbb{C}^*$-zero set of $\vec{f}_{C_4}$ for any $\lambda \in \mathbb{C}^*$. Note that even in the special case of $k_{ij} = 1$, this orbit is *not* balanced (in the sense of [32, Definition 3.1]) for $\lambda \neq 1$.

3. Finally, the third subnetwork $\vec{H}_3$ in Figure 4.2 is balanced if $k_{10}k_{12} = k_{32}k_{03}$. With the choice $\vec{T}_3 < \vec{H}_3$, and following the construction above, we have

$$\boldsymbol{x}(\lambda) = \begin{bmatrix} +k_{01}/2\mathrm{i}\lambda & -k_{01}k_{12}/4\lambda^2 & -k_{03}/2\mathrm{i}\lambda \end{bmatrix} \quad \text{if} \quad k_{10}k_{12} = k_{32}k_{30}$$

whose image is in the $\mathbb{C}^*$-zero set of $\vec{f}_{C_4}$ for any $\lambda \in \mathbb{C}^*$. This orbit is more interesting as all three complex phase variables x_1, x_2, x_3 are nonconstant relative to the reference phase $x_0 = 1$. It is also *not* balanced (in the sense of [32, Definition 3.1]).

Moreover, if the coupling coefficients satisfy the condition $\boldsymbol{k}^2(C_4) = c^2 \cdot \mathbf{1}$ for some $c \in \mathbb{R}$, then the three $\mathbb{C}^*$-orbits contain one-real-dimensional components on the real torus $(S^1)^3$. Indeed, by the restriction $\lambda = \frac{ce^{\mathrm{i}t}}{2\mathrm{i}}$, the three orbits constructed above reduce to parametrized zero sets of one real-dimension inside the real torus. Here, $\operatorname{sgn}(x) \in \{\pm 1\}$ denotes the sign of $x \in \mathbb{R}$. If we define

$$\sigma^{ij}_{\ell m} = \begin{cases} 0 & \text{if } k_{ij}/k_{\ell m} > 0 \\ \pi & \text{otherwise,} \end{cases} \quad \text{and} \quad \sigma^{ij} = \begin{cases} 0 & \text{if } k_{ij} > 0 \\ \pi & \text{otherwise,} \end{cases}$$

then the three potential real orbits can be expressed as

$$(5.2)\quad \begin{cases} \theta_1 = t + \pi + \sigma^{10} \\ \theta_2 = \pi + \sigma^{12}_{10} \\ \theta_3 = t + \sigma^{30} \end{cases} \text{if } \frac{k_{10}k_{32}}{k_{12}k_{30}} = 1, \quad \begin{cases} \theta_1 = \sigma^{10} + t \\ \theta_2 = \pi + \sigma^{21}_{10} \\ \theta_3 = \sigma^{30} - t \end{cases} \text{if } \frac{k_{10}k_{30}}{k_{12}k_{23}} = 1, \quad \begin{cases} \theta_1 = \sigma^{10} - t \\ \theta_2 = \sigma^{21}_{10} - 2t \\ \theta_3 = \pi + \sigma^{30} - t \end{cases} \text{if } \frac{k_{10}k_{12}}{k_{32}k_{30}} = 1,$$

with $\theta_0 = 0$. It is easy to see if k_{ij}'s are identical, then all three real orbits exist. Moreover, these three orbits intersects at a singular point $(\theta_0, \theta_1, \theta_2, \theta_3) = (0, \pi/2, \pi, \pi/2)$. These positive-dimensional solution sets have been studied in Refs. [28, 32] ([32, Section 5.2], in particular, also provide topological analysis for the orbits). In this example, we showed they can also be derived systematically from balanced subnetworks.

5.2. Multiple even cycles sharing one edge. We now show positive-dimensional zero sets for $\vec{f}_G$ is possible for networks consisting of multiple even cycles sharing a single edge (e.g., Figure 5.1a).

Proposition 5.5. *Suppose G consists of d independent even cycles $C_1, \ldots, C_d$ that share a single edge e, then with the choice of the coupling coefficients*

$$k_{ij} = \begin{cases} sd & \text{if } \{i,j\} = e \\ s & \text{otherwise,} \end{cases}$$

for any $s \in \mathbb{C}^*$, and homogeneous natural frequencies $\vec{w} = w \cdot \vec{1}$ for any $w \in \mathbb{C}^*$, the algebraic Kuramoto system $\vec{f}_G$ derived from the network $(G, K, w \cdot \vec{1})$ has a positive-dimensional $\mathbb{C}^*$-zero set.

Proof. Since G is bipartite, by Theorem 2.3, there exists a facet F of $\check{\nabla}_G$ such that $G_F = G$, and thus $\check{\nabla}_G = \{\vec{0}\} \cup F \cup (-F)$. Consequently,

$$\vec{f}_G(\boldsymbol{x}) = \check{Q}(\vec{G}_F)(\boldsymbol{x}(\lambda)^{\check{Q}(\vec{G}_F)} \circ \boldsymbol{a}(\vec{G}_F))^\top + \check{Q}(\vec{G}_{-F})(\boldsymbol{x}(\lambda)^{\check{Q}(\vec{G}_{-F})} \circ \boldsymbol{a}(\vec{G}_{-F}))^\top.$$

Let T be a spanning tree of G that contains e, and let $\vec{T}$ be the corresponding subdigraph such that $\vec{T} < \vec{G}_F$. We shall arrange of the columns of $\check{Q}(\vec{G}_F)$ and $\check{Q}(\vec{T})$ so that

$$\check{Q}(\vec{G}_F) = \begin{bmatrix} \check{Q}(\vec{T}) & \vec{v}_1 & \cdots & \vec{v}_d \end{bmatrix} \qquad \text{and} \qquad \check{Q}(\vec{T}) = \begin{bmatrix} \check{Q}(\vec{T} - \{e\}) & \vec{v}_e \end{bmatrix}$$

where $\vec{v}_e$ is the incidence vector corresponding to the edge e that is shared by all cycles, and $\vec{v}_1, \ldots, \vec{v}_d$ are the incidence vectors corresponding to the arcs associated with the edges $e_1, \ldots, e_d$ in $\mathcal{E}(C_1) \setminus \mathcal{E}(T), \ldots, \mathcal{E}(C_d) \setminus \mathcal{E}(T)$, respectively. Pick basis vectors $\vec{\eta}_1, \ldots, \vec{\eta}_d$ of $\ker \check{Q}(\vec{G}_F)$ that are

primitive cycle vector associated with the cycles $C_1, \dots, C_d$ with consistent signs (i.e., their shared nonzero entries have the same sign). Let

$$\boldsymbol{\eta} = \begin{bmatrix} \boldsymbol{\eta}_T & u_1 & \cdots & u_d \end{bmatrix} := (\vec{\eta}_1 + \cdots + \vec{\eta}_d)^\top.$$

Then the first $n-1$ entries of $\boldsymbol{\eta}_T$ are ± 1, and its n-th entry is $\pm d$. The signs depend on the choices of the primitive cycle vectors.

With these, we define the nonconstant function $\boldsymbol{x} : \mathbb{C}^* \to (\mathbb{C}^*)^n$, given by

$$\boldsymbol{x}(\lambda) = (\lambda \cdot \boldsymbol{\eta}_{\vec{T}} \circ \boldsymbol{a}(\vec{T})^{-I})^{\check{Q}(\vec{T})^{-1}},$$

as above. Then, as demonstrated in Theorem 4.5,

$$\check{Q}(\vec{G}_F)(\boldsymbol{x}(\lambda)^{\check{Q}(\vec{G}_F)} \circ \boldsymbol{a}(\vec{G}_F))^\top = \vec{0}$$

for any $\lambda \in \mathbb{C}^*$. It remains to show $\check{Q}(\vec{G}_{-F})(\boldsymbol{x}(\lambda)^{\check{Q}(\vec{G}_{-F})} \circ \boldsymbol{a}(\vec{G}_{-F}))^\top$ also vanishes. Since $\boldsymbol{a}(\vec{T}) = \frac{\boldsymbol{k}(\vec{T})}{2\mathfrak{i}} = [\, s \ \cdots \ s \ sd \,]$,

$$\boldsymbol{x}(\lambda)^{-\check{Q}(\vec{T})} \circ \boldsymbol{a}(\vec{T}) = (\lambda \cdot \boldsymbol{\eta}_{\vec{T}} \circ \boldsymbol{a}(\vec{T})^{-I})^{-I} \circ \boldsymbol{a}(\vec{T}) = \lambda^{-1} \boldsymbol{\eta}_{\vec{T}}^{-I} \circ \begin{bmatrix} \frac{s^2}{-4} & \cdots & \frac{s^2}{-4} & \frac{s^2 d^2}{-4} \end{bmatrix} = \frac{s^2}{-4\lambda} \boldsymbol{\eta}_{\vec{T}}$$

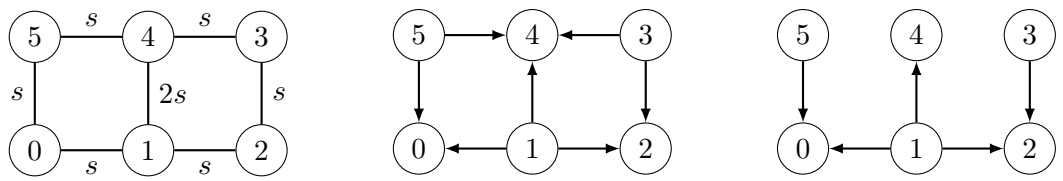
Moreover, by construction, $\boldsymbol{x}(\lambda)^{\vec{v}_i} \cdot \boldsymbol{a}(e_i) = \lambda u_i$ for $i = 1, \dots, d$. Therefore,

$$\boldsymbol{x}(\lambda)^{-\vec{v}_i} \cdot \boldsymbol{a}(e_i) = \lambda^{-1} \boldsymbol{a}^2(e_i) u_i^{-1} = \frac{s^2}{-4\lambda} u_i,$$

since $u_1 \in \{\pm 1\}$. This shows that

$$\boldsymbol{x}(\lambda)^{\check{Q}(\vec{G}_{-F})} \circ \boldsymbol{a}(\vec{G}_{-F}) = \begin{bmatrix} \boldsymbol{x}(\lambda)^{-\check{Q}(\vec{T})} \circ \boldsymbol{a}(\vec{T}) & \boldsymbol{x}(\lambda)^{-\vec{v}_1} \cdot \boldsymbol{a}(e_1) & \boldsymbol{x}(\lambda)^{-\vec{v}_2} \cdot \boldsymbol{a}(e_1) \end{bmatrix} = \frac{s^2}{-4\lambda} \boldsymbol{\eta}.$$

Consequently, $\vec{f}_G(\boldsymbol{x}(\lambda)) = \vec{0}$ for any $\lambda \in \mathbb{C}^*$, i.e., the $\mathbb{C}^*$-zero set of $\vec{f}_G$ is positive-dimensional. ■



(a) G (b) A facet subdigraph $\vec{G}_F$ (c) $\vec{T}$ inside $\vec{G}_F$

Figure 5.1: A network with two independent even cycles.

Example 5.6. Consider the network shown in Figure 5.1a. We fix a particular facet whose facet subdigraph is shown in Figure 5.1b. We also fix a choice of $\vec{T} < \vec{G}_F$ shown in Figure 5.1c. With this choice of $\vec{T}$ and the ordering of the arcs $(1,0)$, $(1,2)$, $(3,2)$, $(5,0)$, $(1,4)$, $(3,4)$, $(5,4)$, we have

$$\check{Q}(\vec{T}) = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \qquad \check{Q}(\vec{T})^{-1} = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \end{bmatrix}, \qquad \vec{\eta}_1 = \begin{bmatrix} 0 \\ -1 \\ +1 \\ 0 \\ +1 \\ -1 \\ 0 \end{bmatrix}, \qquad \vec{\eta}_2 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ +1 \\ +1 \\ 0 \\ -1 \end{bmatrix}$$

as constructed in the proof above. We also define

$$\boldsymbol{\eta} = \vec{\eta}_1^{\top} + \vec{\eta}_2^{\top} = \begin{bmatrix} -1 & -1 & +1 & +1 & +2 & -1 & -1 \end{bmatrix} \quad \text{and} \quad \boldsymbol{\eta}_T = \begin{bmatrix} -1 & -1 & +1 & +1 & +2 \end{bmatrix}.$$

We then verify that the construction $\boldsymbol{x}(\lambda) = (\lambda \cdot \boldsymbol{\eta}_T \circ \boldsymbol{a}(\vec{T})^{-I})^{\check{Q}(\vec{T})^{-1}}$ above produces

$$(\lambda \begin{bmatrix} -1 & -1 & +1 & +1 & +2 \end{bmatrix} \circ \begin{bmatrix} \frac{s}{2\mathrm{i}} & \frac{s}{2\mathrm{i}} & \frac{s}{2\mathrm{i}} & \frac{s}{2\mathrm{i}} & \frac{2s}{2\mathrm{i}} \end{bmatrix}^{-I})^{\check{Q}(\vec{T})^{-1}} = \begin{bmatrix} -\frac{2\mathrm{i}\lambda}{s} & +1 & +\frac{2\mathrm{i}\lambda}{s} & -1 & +\frac{2\mathrm{i}\lambda}{s} \end{bmatrix},$$

which parametrizes an one-dimensional $\mathbb{C}^*$-zero set of $\vec{f}_G = 0$. Moreover, by choosing $\lambda(t) = \frac{se^{\mathrm{i}t}}{2\mathrm{i}}$, $\boldsymbol{x} \in (S^1)^5$ for any $t \in \mathbb{R}$. Therefore, $\log(\boldsymbol{x})$ produces the one-dimensional real zero set

$$(\theta_0, \theta_1, \theta_2, \theta_3, \theta_4, \theta_5) = (0, t + \pi, 0, t, \pi, t)$$

for the transcendental Kuramoto system (2.2) derived from the network in Figure 5.1a.

6. Concluding remarks. We studied the structure of the zero sets of the Kuramoto equations and its algebraic counterpart. By leveraging a recently discovered three-way connection between the graph-theoretic, convex-geometric, and tropical view points, we answered four key questions.

1. For Question 1.1, we showed that *for generic natural frequencies and generic but symmetric coupling coefficients, the $\mathbb{C}^*$ root count for the algebraic Kuramoto system coincides with the adjacency polytope bound, and as a corollary we showed that the algebraic Kuramoto system is Bernshtein-general.*
2. For Question 1.2, we provided a description of the *exceptional coupling coefficients for which the $\mathbb{C}^*$ root count of the algebraic Kuramoto system drops below the generic zero count.* This description used graph-theoretic, combinatorial and toric information.
3. For Question 1.3, we analyzed unicycle networks and developed a *full stratification of the coupling coefficient space and computed the $\mathbb{C}^*$ root count over each stratum.*
4. For Question 1.4, we established sufficient conditions on unicycle networks and networks consisting of cycles sharing a single edge under which *there will be non-isolated real and complex zero sets for Kuramoto systems* through explicit constructions.

While the analysis for the last two questions required some topological restrictions, it appears hopeful that the approach taken here can be generalized to other networks. We hope our work will spark interest in the full analysis of the typical and atypical solutions to Kuramoto equations.

The proofs are constructive. In particular, the homotopies used in Theorem 3.5 and Lemma 4.16 are specialized polyhedral homotopy [23] that will likely bring significant performance improvement.

Finally, we speculate that the explicit connection between the root count of the algebraic Kuramoto equations and the normalized volume of adjacency polytopes may also allow algebraic geometers to directly contribute to the geometric study of adjacency polytopes.

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Appendix A. Notations. Here, we list notations used in this paper that may not be standard.

$\mathbb{C}\{\tau\}$ The field of Puiseux series in τ with complex coefficients is denoted $\mathbb{C}\{\tau\}$, and only convergent series, representing germs of one-dimensional analytic varieties, will be relevant.

$\mathcal{C}(P, \Delta_\omega)$ The (closed) secondary cone of a regular subdivision Δ_ω in a polytope P.

$\Sigma_\omega(P)$ The regular subdivision of a point configuration P induced by a lifting function $\omega : P \to \mathbb{Q}$.

$\check{\nabla}_G$ The point configuration associated with the adjacency polytope derived from a connected graph G. It is defined to be $\{\pm(\vec{e}_i - \vec{e}_j) \mid \{i, j\} \in \mathcal{E}(G)\} \cup \{\vec{0}\}$.

$\check{\nabla}_G^\omega$ "Lifted" version of point configuration $\check{\nabla}_G$ induced by a lifting function $\omega : \check{\nabla}_G \to \mathbb{Q}$. It consists of the points $(\vec{p}, \omega(\vec{p})) \subset \mathbb{R}^{n+1}$ for all $\vec{p} \in \check{\nabla}_G$.

$\vec{f}_{(G,K,\vec{w})}, \vec{f}_{(G,K)}, \vec{f}_G$ The algebraic Kuramoto system derived from a Kuramoto network $(G, K, \vec{w})$. If the choice of the natural frequencies $\vec{w}$ is not relevant to the discussions (or assumed to be generic), the notation $\vec{f}_{(G,K)}$ will be used. Similarly, if only graph topology is of relevance, we will simply use $\vec{f}_G$.

$\vec{f}_G^*$ The randomized algebraic Kuramoto system, i.e., $R \cdot \vec{f}_G$ for a generic square matrix R.

$(G, K, \vec{w}), (G, K)$ A Kuramoto network. G is the underlying graph, $K = \{k_{ij} \mid \{i, j\} \in \mathcal{E}(G)\}$ with $k_{ij} = k_{ji}$ encodes the coupling coefficients, and $\vec{w} = (w_0, \ldots, w_n)^\top$ contains the natural frequencies.

$\mathrm{init}_{\boldsymbol{v}}(f)$ For a Laurent polynomial f in $x_1, \ldots, x_n$, the *initial form* of f with respect to a vector $\boldsymbol{v}$ is the Laurent polynomial $\mathrm{init}_{\boldsymbol{v}}(f)(\boldsymbol{x}) := \sum_{\vec{a} \in (S)_{\vec{v}}} c_{\vec{a}}\, \boldsymbol{x}^{\vec{a}}$, where $(S)_{\vec{v}}$ is the subset of S on which $\langle\, \boldsymbol{v}\,, \cdot\, \rangle$ is minimized. It is denoted $\mathrm{init}_{\boldsymbol{v}}(f)$. For a system $\vec{f} = (f_1, \ldots, f_q)$ of Laurent polynomials, $\mathrm{init}_{\boldsymbol{v}}(\vec{f}) = (\mathrm{init}_{\boldsymbol{v}}(f_1), \ldots, \mathrm{init}_{\boldsymbol{v}}(f_q))$.

$Q(\vec{H}), \check{Q}(\vec{H})$ For a digraph $\vec{H}$, its *incidence matrix* $Q(\vec{H})$ is the matrix with columns $\vec{e}_i - \vec{e}_j$ such that $(i, j) \in \mathcal{E}(\vec{G}_F)$. Since we set $\vec{e}_0$ to be the zero vector, the first row is all zeros. Therefore, we instead consider the *reduced incidence matrix*, $\check{Q}(\vec{H})$, with $n = |\mathcal{V}(\vec{H})| - 1$ rows, which is the incidence matrix of $\vec{H}$ with the first row deleted. The ordering of the columns in both is arbitrary, but, when appears in the same context with other incidence vectors, a consistent ordering is assumed. Here, the adjective "reduced" emphasize the fact that the labels for the nodes in the graph are $0, 1, \ldots$, and therefore, for a digraph $\vec{H}$ of $n+1$ nodes, $\check{Q}(\vec{H})$ only has n rows.

$\mathrm{MV}(P_1, \ldots, P_n)$ Given n convex polytopes $P_1, \ldots, P_n \subset \mathbb{R}^n$, the *mixed volume* of $P_1, \ldots P_n$ is the coefficient of the monomial $\lambda_1 \cdots \lambda_n$ in the homogeneous polynomial $\mathrm{vol}_n(\lambda_1 P_1 + \ldots + \lambda_n P_n)$ where $P + Q = \{p + q \,:\, p \in P,\ q \in Q\}$ denotes the Minkowski sum and vol_n is the standard $n-$dimensional Euclidean volume form.

$\mathcal{N}_G(i), \mathcal{N}^+_{\vec{G}}(i), \mathcal{N}^-_{\vec{G}}(i)$ For a graph G and a node i of G, $\mathcal{N}_G(i)$ is the set of nodes that are adjacent to i. Similarly, the other two are the adjacent nodes through outgoing and incoming arcs in a digraph, respectively.

Vol The normalized volume of a set in $\mathbb{R}^n$, which is defined to be $n!$ times the Euclidean volume form. The usage is restricted to convex polytopes here, and we adopt the convention that $\mathrm{Vol}(X) = 0$, if is not full dimensional.

Appendix B. Elementary lemmata.

We restate Lemma 3.3 and provide an elementary proof.

Lemma B.1. *For a generic symmetric lifting function ω for $\check{\nabla}_G$, let Δ be a simplex in Δ_ω. Then the digraph $\vec{G}_\Delta$ is acyclic, and its underlying graph G_Δ is a spanning tree of G.*

Proof. Let $(\vec{\alpha}, 1)$ be the upward pointing inner normal that defines the cell Δ as a projection of a lower facet of $\check{\nabla}_G^\omega$. Suppose $\vec{G}_\Delta$ has a simple directed cycle $i_1 \to \cdots \to i_m \to i_1$. Then

$$\langle\, \vec{\alpha}\,, \vec{e}_{i_k} - \vec{e}_{i_{k+1}} \,\rangle + 1 + \delta_{i_k, i_{k+1}} = 0 \quad \text{for } k = 1, \ldots, m,$$

where $i_{m+1} = i_1$. Summing these m equations produces $m + \sum_{k=1}^m \delta_{i_k, i_{k+1}} = 0$, which is not possible under the assumption that δ_{ij} are sufficiently close to 0. So $\vec{G}_\Delta$ must be acyclic.

Moreover, $\dim(\Delta) = n$, by assumption. That is, $\{\vec{e}_i - \vec{e}_j \mid (i, j) \in \vec{G}_\Delta\}$ is must span $\mathbb{R}^n$ as a set of vectors (since $\vec{0} \in \Delta$, by construction). Therefore, for every $i \in \{0, \ldots, n\}$, either $\vec{e}_i - \vec{e}_j \in \Delta$ or $\vec{e}_j - \vec{e}_i \in \Delta$ for some $j \in \{0, \ldots, n\} \setminus \{i\}$. That is, G_Δ must be spanning. ∎

Lemma B.2. *Recall the set up in Example 4.18. For each stratification of the coupling coefficients, there exist parameter values where all $\mathbb{C}^*$ solutions are real.*

Proof. Recall the stratification of the coupling coefficients for C_4 given in Example 4.18. By Proposition 4.17, we have an upper bound on the number of $\mathbb{C}^*$ (and therefore $\mathbb{R}^*$) solutions in each case. Using the parameter values given in Example 4.18, we use `HomotopyContinuation.jl` to find all complex solutions and certify they are real using interval arithmetic [7, 6]. Since the upper bound in Proposition 4.17 matches the number of certified complex solutions, the result follows. ■

Appendix C. Regular zeros and the principle of homotopy continuation.

An isolated zero $\boldsymbol{x} \in (\mathbb{C}^*)^n$ of a square Laurent system $\vec{f}$ is said to be *regular* if the Jacobian matrix $D\vec{f}$ is invertible at $\boldsymbol{x}$. Otherwise, it is *singular*. This definition applies to both the interpretation $\vec{f} : (\mathbb{C}^*)^n \to (\mathbb{C}^*)^n$, as a holomorphic function and the interpretation $\vec{f} : ((\mathbb{R}^*)^2 \setminus \{0\})^n \to \mathbb{R}^{2n}$, as a smooth function between real manifolds. Sard's Theorem states that for almost all choices of $\vec{\epsilon}$, all zeros of the square system $\vec{f}(\boldsymbol{x}) - \vec{\epsilon}$ are regular. In other words, having regular zeros is a "generic" behavior.

In the arguments presented in this paper, we made frequent reference of a collection of theorems from differential geometry that we simply referred to as the *principle of homotopy continuation*. This is a deep theoretical framework that has found many important applications. Here, we give an overly simplified description. Consider a smooth function $H : M \times \mathbb{R} \to N$ where M and N are manifolds of the same dimension. Suppose $\boldsymbol{x}_0 \in M$ is a regular zero of $H(\cdot, t_0)$ for some $t_0 \in \mathbb{R}$, then there exists a smooth function $\boldsymbol{x}(t) : [t_0, t_1) \to M$, for some $t_1 > t_0$, such that $\boldsymbol{x}(t_0) = \boldsymbol{x}_0$ and $H(\boldsymbol{x}(t), t) = \vec{0}$. There are exactly three possibilities for the domain $[t_0, t_1)$ of $\boldsymbol{x}(t)$: (A) $t_1 = \infty$; (B) $\boldsymbol{x}(t_1)$ is a singular zero of $H(\cdot, t_0)$; or (C) $\boldsymbol{x}(t)$ converges to a limit point outside M as $t \to t_1$. This can be established through repeated applications of Inverse Function Theorem and Implicit Function Theorem. Its complex version can be established as a special case of analytic continuations.

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