A toric deformation method for solving Kuramoto equations*

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Abstract. The study of frequency synchronization configurations in Kuramoto models is a ubiquitous mathematical problem that has found applications in many seemingly independent fields. In this paper, we focus on networks whose underlying graph are cycle graphs. Based on the recent result on the upper bound of the frequency synchronization configurations in this context, we propose a toric deformation homotopy method for locating all frequency synchronization configurations with complexity that is linear in this upper bound. Loosely based on the polyhedral homotopy method, this homotopy induces a deformation of the set of the synchronization configurations into a series of toric varieties, yet our method has the distinct advantage of avoiding the costly step of computing mixed cells. We also explore the important consequences of this homotopy method in the context of direct acyclic decomposition of Kuramoto networks and tropical stable intersection points for Kuramoto equations.

Key words. Toric varieties, Kuramoto model, Kuramoto equations, adjacency polytope, polyhedral homotopy, tropical geometry

1. Introduction.

A network of connected oscillators is set of objects, varying between two states, that can influence one another through connections among the oscillators. A network of $N = n + 1$ oscillators can be mathematically described by a weighted graph $G = (V, E, A)$ on the vertex set $V = \{0, \dots, n\}$, representing the oscillators, with edges E representing the connections among the oscillators, and weights $A = \{a_{ij}\}$ on the edges representing the *coupling strength* along edge $\{i, j\}$, i.e., the level of mutual influence between oscillators i and j. For each $i = 0, \dots, n$, oscillator i has its own natural frequency ω_i, a constant such that $\frac{d\theta_i}{dt} = \omega_i$ if the oscillator is an isolated vertex of the network. Of course, in practice, an oscillator is affected by those it is adjacent to, and describing the full network can be done using the Kuramoto model [15]

$$\frac{d\theta_i}{dt} = \omega_i - \sum_{j \in \mathcal{N}_G(i)} a_{ij} \sin(\theta_i - \theta_j), \qquad \text{for } i = 0, \dots, n, \tag{1.1}$$

where each $\theta_i \in [0, 2\pi)$ is the phase angle that describes the status of the i-th oscillator, and $\mathcal{N}_G(i)$ is the set of neighbors of the i-th vertex. A fundamental mathematical problem in the study of Kuramoto model as well as the behavior of coupled oscillators is the occurrence of *synchronization*. Among many different notions of synchronization, this paper focuses only on "frequency synchronization". *Frequency synchronization* occurs when the two competing forces of oscillators to stay with their natural frequency and the influence of their neighbors

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reach equilibrium for all oscillators and they are all tuned to the same frequency. More precisely, a configuration $\boldsymbol{\theta} = (\theta_0, \ldots, \theta_n)$ is said to be in *frequency synchronization* if at this point $\frac{d\theta_i}{dt} = c$ for a common constant c for all i. That is,

$$\omega_i - \sum_{j \in \mathcal{N}_G(i)} a_{ij} \sin(\theta_i - \theta_j) = c, \qquad \text{for } i = 0, \ldots, n, \tag{1.2}$$

in the variables $\theta_1, \ldots, \theta_N$ with c being some constant.

In this paper, we focus on the cases where the underlying graph is a cycle, i.e., the set of edges E of G consists of $\{0,1\}, \{1,2\}, \ldots, \{n-1, n\}, \{n, 0\}$. In [5], an upper bound on the total number of isolated solutions the *synchronization equations* (1.2) has is shown to be $N\binom{N-1}{\lfloor (N-1)/2 \rfloor}$ using the theory of birationally invariant intersection index. Indeed, this upper bound is generically sharp for a complexified version of (1.2). It is then natural to ask if there exists an algorithm that can locate all solutions of (1.2) with a complexity that is linear in this solution bound $N\binom{N-1}{\lfloor (N-1)/2 \rfloor}$. This is the main topic that this paper addresses.

The primary contribution of this paper is to provide a homotopy method in the spirit of polyhedral homotopy that will find *all* isolated solutions of (1.2). The total number of homotopy paths to be tracked with this method is exactly the solution bound $N\binom{N-1}{\lfloor (N-1)/2 \rfloor}$. Yet, this method offers significant advantages over a direct application of polyhedral homotopy via the following features:

- our method does not require the costly step of computing mixed cells and mixed volume;
- our method uses integer liftings of $\{0, 1, 2\}$ and hence avoids the well known numerical instability caused by random liftings; and

The secondary contribution is an explicit description of a regular unimodular triangulation of the adjacency polytope which significantly strengthens the previous volume and facet description results [5]. The tertiary contribution is our significant refinement for the direct acyclic decomposition scheme proposed in [2] for cycle graphs. This refined scheme is capable of reducing a network into simplest subnetworks known as primitive subnetworks for which frequency synchronization configurations can be computed directly and efficiently. Finally, we provide an interpretation of our result in terms of tropical algebraic geometry as well as the equivalence of three rather different perspective to the Kuramoto equations.

The paper is organized as follows. In section 2 we briefly review the Kuramoto model and Kuramoto equations. Then section 3 describes a complex algebraic formulation of the Kuramoto equations as a system of rational equations over the complex algebraic torus $(\mathbb{C}^*)^n$. Recent results on the generic root count of the algebraic Kuramoto equations, known as the adjacency polytope bound, is reviewed in section 4. We expand on this result by describing explicit formula for a regular unimodular triangulation of the adjacency polytope in section 5. The resulting algorithm is outlined in section 6, and section 7 describe the software implementation. Basing on this triangulation, we develop our homotopy method in section 8. In section subsection 9.1 and subsection 9.2 we interpret our results in the broader context, and the conclusion follows in section 10.

2. Kuramoto model and Kuramoto equations.

A simple mechanical analog of the coupled oscillator model (1.1) is a spring network, shown in Figure 1, that consists of a set of

weightless particles constrained to move on the unit circle without friction or collision [8]. Real numbers $a_{ij} = a_{ji}$ characterizing the stiffness of the springs connecting particles i and j are known as *coupling strength*, and $\frac{d\theta_i}{dt}$ represents the angular velocity of the i-th particle. An important class of special configurations in which the angular velocity of *all* particles can become perfectly aligned are known as *frequency synchronization*. That is, $\frac{d\theta_i}{dt} = c$ for $i = 0, \ldots, n$ and a constant c. By adopting a rotational frame of reference, we can always assume $c = 0$. That is, frequency synchronization configurations are equivalent to equilibria of the ordinary differential equations (1.1). As a result of the symmetry assumption that $a_{ij} = a_{ji}$, the $n+1$ equilibrium equations must sum to zero. This allows the elimination of one of the equations, producing the system of n equations in n unknowns

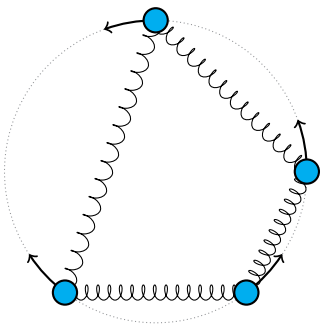

Figure 1. *A spring network*

$$\omega_i - \sum_{j \in \mathcal{N}_G(i)} a_{ij} \sin(\theta_i - \theta_j) = c, \qquad \text{for } i = 1, \ldots, n.$$

Despite its mechanical origin, the above frequency synchronization system naturally appears in a long list of seemingly unrelated fields, including chemistry, electrical engineering, biology, and computer security. We refer to [8] for a detailed list.

3. Complex algebraic formulation of Kuramoto equations. With proper complexification, the Kuramoto equations (1.2) can be reformulated as a system of Laurent polynomials over the algebraic torus $(\mathbb{C}^*)^N$. This form is crucial in applying the homotopy continuation theory. Using the identity $\sin(\theta_i - \theta_j) = \frac{1}{2\mathbf{i}}(e^{\mathbf{i}(\theta_i - \theta_j)} - e^{-\mathbf{i}(\theta_i - \theta_j)})$ where $\mathbf{i} = \sqrt{-1}$, (1.2) can be transformed into

$$c_i - \sum_{j \in \mathcal{N}_G(i)} \frac{a_{i,j}}{2\mathbf{i}} (e^{\mathbf{i}\theta_i} e^{-\mathbf{i}\theta_j} - e^{-\mathbf{i}\theta_i} e^{\mathbf{i}\theta_j}) = 0 \qquad \text{for } i = 1, \ldots, N.$$

With the substitution $x_i := e^{\mathbf{i}\theta_i}$ for $i = 1, \ldots, n$, we obtain the system of rational equations

$$F_{G,i}(x_1, \ldots, x_n) = c_i - \sum_{j \in \mathcal{N}_G(i)} a'_{ij} \left(\frac{x_i}{x_j} - \frac{x_j}{x_i} \right) = 0 \qquad \text{for } i = 1, \ldots, n \tag{3.1}$$

where $a'_{ij} = \frac{a_{ij}}{2\mathbf{i}}$ and $x_0 = 1$ is a constant. This system, $F_G = (F_{G,1}, \ldots, F_{G,n})$, is a system of n Laurent polynomial equations in the n complex variables $\mathbf{x} = (x_1, \ldots, x_n)$. In the following, it will be referred to as the (algebraic) system of *synchronization equations* for a Kuramoto model, or simply a *synchronization system*. Since x_i's appear in the denominator positions, F_G is only defined on the algebraic torus $(\mathbb{C}^*)^n = (\mathbb{C} \setminus \{0\})^n$. Clearly, each equivalence class of real solutions of (1.2) (modulo translations by multiples of 2π) corresponds to a single solution of (3.1) in $(\mathbb{C}^*)^n$.

If we consider F_G as a column vector, then for any nonsingular $n \times n$ matrix R, the two systems $R \cdot F_G = \mathbf{0}$ and $F_G = \mathbf{0}$ have the exact same solution set. Therefore in the following

we focus on the system

$$F_G^R = R \cdot F_G = \begin{bmatrix} r_{11} & \cdots & r_{1n} \\ \vdots & \ddots & \vdots \\ r_{n1} & \cdots & r_{nn} \end{bmatrix} \begin{bmatrix} F_{G,1} \\ \vdots \\ F_{G,n} \end{bmatrix}$$

It is easy to verify that for generic choices of the matrix R, there is no complete cancellation of the terms, and thus F_G^R is an (unmixed) system of the form

$$F_{G,k}^R = c_k^R - \sum_{\{i,j\} \in \mathcal{E}(G)} a_{ijk}^R \left(\frac{x_i}{x_j} - \frac{x_j}{x_i} \right) \quad \text{for } k = 1, \ldots, n \tag{3.2}$$

where c_k^R and a_{ijk}^R are the resulting coefficients after collection of similar terms, and none of them are zero. This system will be referred to as the *unmixed form* of the synchronization equations, and it will be the main focus of the rest of this paper.

4. Maximum and Generic Root Count. In this section we briefly review the existing results on the generic root count of (1.2), (3.1), and (3.2).

In [1], an upper bound on the number of equilibria of the Kuramoto model (solutions to (1.2)) induced by a graph of N vertices with any coupling strengths is shown to be $\binom{2N-2}{N-1}$. This bound can be understood as a bi-homogeneous Bézout number on the algebraic version (3.1) or (3.2): Via the map $y_i = x_i^{-1}$, the two systems can be translated into equivalent systems that have a bi-degree of $(1, 1)$ with respect to the partition and are defined in $((x_1, \ldots, x_n), (y_1, \ldots, y_n))$ with the additional conditions that $x_i y_i = 1$ for $i = 1, \ldots, n$. It is easy to verify that the bi-homogeneous Bézout number will be $\binom{2n}{n} = \binom{2(N-1)}{N-1}$.

Recent studies (e.g. [6]) suggest tighter bounds on the number of isolated complex solutions may exist when the network is sparsely connected. In the cases where the underlying graph is a cycle, a sharp bound is established in [5] using the theory of the birationally invariant intersection index as well as a construction known as the *adjacency polytope bound* which we shall review briefly here.

A polytope is a bounded intersection of finitely many closed half-spaces. The adjacency polytope is a polytope constructed to encode the topological information of the synchronization system (3.2). Given an undirected graph G with edge set $\mathcal{E}(G)$, its *adjacency polytope* is defined to be

$$P_G = \operatorname{conv} \{ \mathbf{e}_i - \mathbf{e}_j \mid \{i, j\} \in \mathcal{E}(G) \} \tag{4.1}$$

where we adopt the convention that $\mathbf{e}_0 = \mathbf{e}_{n+1} = \mathbf{0}$. That is, the adjacency polytope of G is the convex hull of a set of line segments, each corresponding to an edge in G. P_G is a *lattice polytope* in the sense that all its vertices have integer coordinates.

The *adjacency polytope bound* [3] of a Kuramoto system (1.2) on the graph G is defined to be $n! \operatorname{vol}(P_G)$, the *normalized volume* of P_G. This bound is an upper bound for the number of isolated complex solutions for the systems (3.2) and (3.1). Consequently, it is also an upper bound for the number of real solutions that the original synchronization system (1.2) has.

In the case of a cycle graph of N nodes, i.e.,

$$G = C_N = (\{0, \ldots, N-1\}, \{\{0,1\}, \ldots, \{N-2, N-1\}, \{N-1, 0\}\}),$$

the recent paper [5] established the explicit formula $N\binom{N-1}{\lfloor (N-1)/2 \rfloor}$ for the adjacency polytope bound. Furthermore it is shown that this bound coincides with the birationally invariant intersection index in $(\mathbb{C}^*)^n$ of the Kuramoto system (3.1) as a member of a family of rational functions. In this paper, we strengthen this result by producing an explicit construction of a unimodular triangulation of the adjacency polytope for cycle graphs and define a homotopy method base on this triangulation.

Remark 4.1. The theory of birationally invariant intersection index [13, 14] (as well as the general intersection theory [9] and homotopy continuation theory [16, 19]) shows that the adjacency polytope bound is "generically exact" in the sense that if one chooses the coefficients of the algebraic Kuramoto equations (3.1) randomly then, with probably one, the total number of isolated complex solutions that system has is exactly the adjacency polytope bound $N\binom{N-1}{\lfloor (N-1)/2 \rfloor}$.

Stated more precisely, there exists a nonzero polynomial D whose variables are the coefficients $\{\omega_i\}$ and $\{a_{ij}\}$ of (3.2) such that for all choices of $\omega_i\}$ and $\{a'_{ij}\}$ where $D \neq 0$, the total number of isolated complex roots of (3.1) are all nonsingular reaches the adjacency polytope bound.

5. A Regular, Unimodular Triangulation of the Adjacency Polytope. A *subdivision* of an n-dimensional polytope P is a collection of polytopes $P_1, \ldots, P_k \subseteq P$ such that

1. $\dim P_i = n$ for all i,
2. $P_i \cap P_j$ is either empty or a face common to both P_i and P_j, and
3. $P = \cup_i P_i$.

A *triangulation*, a.k.a. *simplicial subdivision* of a polytope is a subdivision consisting of simplices. Furthermore, a triangulation is said to be *unimodular* if all the member simplices are lattice simplices of normalized volume 1.

In order to be used in our homotopy construction, the "regularity" property of the triangulation is also required. A triangulation of a polytope is said to be *regular* if it is the projection of the lower facets of a lifting of the polytope into one-higher dimension. Stated more precisely, given a polytope $P = \operatorname{conv}\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ in $\mathbb{R}^n$ and and weights $\omega_1, \ldots, \omega_m \in \mathbb{R}$, the new polytope

$$P' = \operatorname{conv}\{(\mathbf{v}_i, \omega_i) \in \mathbb{R}^{n+1} \mid i = 1, \ldots, m\}$$

is a lifting of P into one-higher dimension. The projections of *lower facets*, that is, the facets whose inner normal vectors have positive last entry, to the first n coordinates is called a *regular subdivision*, or a *regular triangulation* if all facets are simplices.

For the cycle graph C_N on $N = n+1$ nodes, we will construct a unimodular triangulation for the adjacency polytope P_{C_N} by finding and triangulating all of its *facets*: the faces of codimension 1. Using the set of facets $\mathcal{F}(P_{C_N})$, a well known subdivision of P_{C_N} can be constructed as the set of pyramids formed by the facets and a fixed interior point as the common apex. That is, fixing any interior point $\mathbf{p} \in P_{C_N}$, the set

$$\{\operatorname{conv} F \cup \{\mathbf{p}\} \mid F \in \mathcal{F}(P_{C_N})\}$$

forms a subdivision of P_{C_N}. By further triangulating each facet, the above subdivision can be refined into a triangulation of P_{C_N}. That is, if $T(F)$ is a triangulation of the face $F \in \mathcal{F}(P_{C_N})$ then the set

$$\{\operatorname{conv} C \cup \{\mathbf{p}\} \mid C \in T(F), F \in \mathcal{F}(P_{C_N})\}$$

for a fixed interior point $\mathbf{p}$ form a triangulation of P_{C_N}. This is the strategy that we will follow in this section. The choice of the interior point $\mathbf{p}$ will be the origin $\mathbf{0}$ which is an interior point of P_{C_N} since it is the average of $\mathbf{e}_i - \mathbf{e}_j$ and $\mathbf{e}_j - \mathbf{e}_i$ for all edges $\{i, j\}$.

It was shown in [5] that P_{C_N} is unimodularly equivalent to the polytope

$$Q_N = \operatorname{conv}\{\pm\mathbf{e}_1, \ldots, \pm\mathbf{e}_n, \pm(\mathbf{e}_1 + \cdots + \mathbf{e}_n)\}$$

via the map $x \mapsto Ax$, where A is the $n \times n$ matrix with 1 on and below the diagonal and 0 everywhere else. Then [5, Proposition 12] and [18, Remark 4.3] identify the facets of Q_N. The geometric structure of this polytope depends on the parity of N. When N is even, the facets can be indexed by the set

$$\Lambda_N = \left\{ (\lambda_1, \ldots, \lambda_N) \in \{-1, 1\}^N \mid \sum_{i=1}^{N} \lambda_i = 0 \right\}$$

and are of the form

$$F_{\boldsymbol{\lambda}} = \operatorname{conv}\{\lambda_1(-\mathbf{e}_1 - \mathbf{e}_2 - \cdots - \mathbf{e}_n), \lambda_2\mathbf{e}_1, \ldots, \lambda_N\mathbf{e}_n, \mid \boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_N) \in \Lambda_N\}.$$

When N is odd, we define Λ_N differently: in this case, the facets can be indexed by $\Lambda_N := \cup_{j=1}^{N} \Lambda_{j,N}$ where

$$\Lambda_{j,N} = \left\{ (\lambda_1, \ldots, \lambda_N) \mid \lambda_j = 0,\ \lambda_i \in \{-1, 1\} \text{ for all } i \neq j, \text{ and } \sum_{i=1}^{N} \lambda_i = 0 \right\},$$

and the facet corresponding to $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_N) \in \Lambda_{j,N}$ is given by

$$F_{\boldsymbol{\lambda}} = \operatorname{conv}\{\lambda_1(-\mathbf{e}_1 - \mathbf{e}_2 - \cdots - \mathbf{e}_n), \lambda_2\mathbf{e}_1, \ldots, \widehat{\lambda_j\mathbf{e}_{j-1}}, \ldots, \lambda_N\mathbf{e}_n\}.$$

Here, the notation $\widehat{\lambda_j\mathbf{e}_{j-1}}$ indicates that element is removed from the list.

From the above constructions, we can see that Q_N is *simplicial* (i.e., all the facets are simplices) when N is odd, but is not simplicial when N is even. Via the unimodular equivalence between Q_N and P_{C_N} we have same characterization of the facets of P_{C_N}. As a result of this dichotomy, the construction of the triangulation in the even and odd N cases require very different procedures.

Remark 5.1 (Unimodular equivalence of facets). Another important property worth noting is that the facets of Q_N are all unimodularly equivalent to each other. To see this suppose $F_{\boldsymbol{\lambda}}, F_{\boldsymbol{\lambda}'}$ are facets of Q_N. Then, $F_{\boldsymbol{\lambda}'} = f(F_{\boldsymbol{\lambda}})$ where $f(x) = B_{\boldsymbol{\lambda},\boldsymbol{\lambda}'}x$ and $B_{\boldsymbol{\lambda},\boldsymbol{\lambda}'}$ is the $n \times n$ matrix constructed as follows: first let $\ell = \lambda_1\lambda_1'$. For $1 \leq i \leq n$, note that there is a unique

j such that λ_{i+1} is the j^{th} instance of -1 or the j^{th} instance of 1 in $(\lambda_2, \ldots, \lambda_N)$. Let $\ell\lambda'_{k+1}$ be the j^{th} instance of $\ell\lambda_{i+1}$ in $(\ell\lambda'_2, \ldots, \ell\lambda'_N)$. Set row i of $B_{\boldsymbol{\lambda},\boldsymbol{\lambda}'}$ to be ℓe_k. As a result, $B_{\boldsymbol{\lambda},\boldsymbol{\lambda}'}$ is a permutation matrix (up to simultaneous scaling of all entries by -1), so $\det B_{\lambda,\lambda'} = \pm 1$, hence f yields a unimodular equivalence.

Consider, for example, a case with $N = 3$ and the choices of $\boldsymbol{\lambda} = (1, -1, -1, 1)$ and $\boldsymbol{\lambda}' = (-1, 1, -1, 1)$. In this case, $\ell = \lambda_1\lambda'_1 = -1$. Note that λ_2 is the first instance of -1 in $(\lambda_2, \lambda_3, \lambda_4)$. Now, $-\lambda'_2$ is the first occurrence of $-\lambda_2$ in $(-\lambda'_2, -\lambda'_3, -\lambda'_4)$. So, the first row of $B_{\boldsymbol{\lambda},\boldsymbol{\lambda}'}$ is $-\mathbf{e}_1$. Next, λ_3 is the second occurrence of -1 in $(\lambda_2, \lambda_3, \lambda_4)$, and $-\lambda'_4$ is the second occurrence of -1 in $(-\lambda'_2, -\lambda'_3, -\lambda'_4)$, so the second row of $B_{\boldsymbol{\lambda},\boldsymbol{\lambda}'}$ is $-\mathbf{e}_3$. Since λ_3 is the first occurrence of 1 in $(\lambda_2, \lambda_3, \lambda_4)$, and $-\lambda'_3$ is the first occurrence of 1 in $(-\lambda'_2, -\lambda'_3, -\lambda'_4)$, we have that the third row of $B_{\lambda,\lambda'}$ is $-\mathbf{e}_2$:

$$B_{\boldsymbol{\lambda},\boldsymbol{\lambda}'} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}.$$

Remark 5.2 (Point configuration). In order to be used in a homotopy construction, a stronger triangulation is needed. Define the point set

$$S_{C_N} = \{\mathbf{0}\} \cup \{\mathbf{e}_i - \mathbf{e}_j \mid \{i, j\} \in \mathcal{E}(C_N)\}$$

This set is known as the *support* of the unmixed system (3.2) as it collects the exponents (as points) of all the terms appearing in that system. It is easy to see that $P_{C_N} = \operatorname{conv} S_{C_N}$ since $\mathbf{0}$ is an interior point of P_{C_N} (as $\mathbf{0} = \frac{1}{2}(\mathbf{e}_i - \mathbf{e}_j) + \frac{1}{2}(\mathbf{e}_j - \mathbf{e}_i)$). In our constructions, we will require all simplices in a triangulation to have vertices within the set S_{C_N}. This is known as a triangulation of a *point configuration*.

In the rest of this section, we describe the construction of regular unimodular triangulation of P_{C_N} in the cases with even and odd N respectively.

5.1. Even N. For the entirety of this subsection, we assume that N is even. From the preceding discussion, we know that all of the facets of P_{C_N} are unimodularly equivalent due to transitivity of equivalence relations. In particular, all facets of P_{C_N} are unimodularly equivalent to

$$\operatorname{conv}\{\mathbf{e}_0 - \mathbf{e}_1, \mathbf{e}_1 - \mathbf{e}_2, \ldots, \mathbf{e}_{\lfloor \frac{n}{2} \rfloor} - \mathbf{e}_{\lfloor \frac{n}{2} \rfloor + 1}, -(\mathbf{e}_{\lfloor \frac{n}{2} \rfloor + 1} - \mathbf{e}_{\lfloor \frac{n}{2} \rfloor + 2}), \ldots, -(\mathbf{e}_{n-1} - \mathbf{e}_n), -\mathbf{e}_n\}.$$

Let $G_{\boldsymbol{\lambda}}$ denote the facet of P_{C_N} obtained by applying A^{-1} to all points in $F_{\boldsymbol{\lambda}}$. It will be important to keep in mind that $\pm A^{-1}(\mathbf{e}_1 + \cdots + \mathbf{e}_n) = \pm \mathbf{e}_1$. We can then produce a subdivision of P_{C_N} by setting

$$G^{\mathbf{0}}_{\boldsymbol{\lambda}} = \operatorname{conv}\{\mathbf{0}, G_{\boldsymbol{\lambda}}\}.$$

and ranging over all $\boldsymbol{\lambda} \in \Lambda_N$.

To aid us in what follows, we establish the following lemma. Recall that in $\mathbb{R}^n$, we use the convention $\mathbf{e}_0 = \mathbf{e}_{n+1} = \mathbf{0}$.

Lemma 5.3. *Let $V_N = \{\mathbf{v}_0, \dots, \mathbf{v}_N\}$ denote the vertices of $G^{\mathbf{0}}_{\boldsymbol{\lambda}}$ such that*

$$\mathbf{v}_i = \begin{cases} \mathbf{0} & \text{if } i = 0, \\ \lambda_i(\mathbf{e}_{i-1} - \mathbf{e}_i) & \text{if } 1 \le i \le N, \end{cases}$$

When N is even, each $G^{\mathbf{0}}_{\boldsymbol{\lambda}}$ has exactly two triangulations:

$$\Delta_+(G^{\mathbf{0}}_{\boldsymbol{\lambda}}) = \{\text{conv}\{V_N \setminus \{\mathbf{v}_i\}\} \mid \lambda_i = \lambda_1\}$$

and

$$\Delta_-(G^{\mathbf{0}}_{\boldsymbol{\lambda}}) = \{\text{conv}\{V_N \setminus \{\mathbf{v}_i\}\} \mid \lambda_i = -\lambda_1\}$$

Moreover, both of these triangulations are regular.

Proof. Let $N = 2k$. Note that for each $\boldsymbol{\lambda} \in \Lambda_N$, $\dim G^{\mathbf{0}}_{\boldsymbol{\lambda}} = n$ and $G^{\mathbf{0}}_{\boldsymbol{\lambda}}$ has $n+2$ vertices. Thus, there is a unique (up to simultaneous scaling of the coefficients) affine dependence of the form

$$\sum_{i=0}^{N} c_i \mathbf{v}_i = \mathbf{0}$$

satisfying $\sum c_j = 0$ with $c_0, \dots, c_{n+1} \in \mathbb{R}$. Without loss of generality, we may choose $c_0 = 0$ and $c_i = \lambda_i k/N$ for $i > 0$.

The desired conclusions for the lemma then follow from [7, Lemma 2.4.2]. Specifically,

$$\Delta_+(G^{\mathbf{0}}_{\boldsymbol{\lambda}}) = \{\text{conv}\{V \setminus \{\mathbf{v}_i\}\} \mid \lambda_i = \lambda_1\}$$

is the triangulation of $G^{\mathbf{0}}_{\boldsymbol{\lambda}}$ corresponding to the height vector $(\omega_0, \dots, \omega_N)$ where

$$\omega_i = \begin{cases} 0 & \text{if } c_i \le 0, \\ 1 & \text{if } c_i > 0 \end{cases}$$

and

$$\Delta_-(G^{\mathbf{0}}_{\boldsymbol{\lambda}}) = \{\text{conv}\{V \setminus \{\mathbf{v}_i\}\} \mid \lambda_i = -\lambda_1\}$$

is the triangulation corresponding to the heights

$$\omega_i = \begin{cases} 0 & \text{if } c_i \ge 0, \\ 1 & \text{if } c_i < 0. \end{cases}$$

∎

We will be concerned with the particular lifting function $\omega : S_{C_N} \to \mathbb{Z}$ given by

$$\omega(\mathbf{a}) = \begin{cases} 0 & \text{if } \mathbf{a} = \mathbf{0} \\ 2 & \text{if } \mathbf{a} = \pm\mathbf{e}_1 \\ 1 & \text{otherwise} \end{cases}$$

as this will induce the desired regular, unimodular triangulation Δ_N of P_{C_N}. To help us with notation, we will define

$$\Omega_\omega(P) = \text{conv}\{\omega(v) \mid v \in P \cap S_{C_N}\}$$

for any polytope P whose vertices are a subset of S_{C_N}. If X is a collection of polytopes whose vertices are subsets of S_{C_N}, then we let $\Omega_\omega(X) = \{\Omega_\omega(P) \mid P \in X\}$.

First, we identify normal vectors of simplices in $\Omega_\omega(\Delta_+(G^{\mathbf{0}}_\lambda))$. Recall that a vector is *upward-pointing* if its final coordinate is positive.

Lemma 5.4. *Let N be even. If $\boldsymbol{\lambda} \in \Lambda_N$ and $\lambda_1 = 1$, then the upward-pointing inner normal vectors for all simplices in $\Omega_\omega(\Delta_+(G^{\mathbf{0}}_{\boldsymbol{\lambda}}))$ are $x_{\boldsymbol{\lambda}} = (x_1, \ldots, x_{n+1})$ where*

$$x_i = \begin{cases} \sum_{j=1}^{i} \lambda_j & \text{if } 1 \leq i < n+1, \\ 1 & \text{if } i = n+1 \end{cases}$$

as well as

$$y_{\boldsymbol{\lambda},j} = x + \sum_{k=1}^{j-1} \mathbf{e}_k$$

for each $j > 1$ such that $\lambda_j = 1$. If $\lambda_1 = -1$, then the upward-pointing inner normal vectors are $x_\lambda = \sigma(x_{-\boldsymbol{\lambda}}), y_{\boldsymbol{\lambda},j} = \sigma(y_{-\boldsymbol{\lambda},j})$ where $\lambda_j > 0$ and where σ is the map that negates the first n coordinates.

Proof. First observe that, by construction, each vector under consideration is upward-pointing. Next, let $\lambda_1 = 1$. It is then straightforward to verify that

$$\langle \mathbf{x}, \lambda_j(\mathbf{e}_{j-1} - \mathbf{e}_j) + \mathbf{e}_{n+1} \rangle = -\lambda_j^2 + 1 = 0$$

for all $1 < j \leq n+1$. Following this same process, one may verify that the hyperplane for which $y_{\lambda,j}$ is normal contains all vertices of $G^{\mathbf{0}}_{\boldsymbol{\lambda}}$ except $\lambda_j(\mathbf{e}_{j-1} - \mathbf{e}_j)$.

Finally notice that if $\boldsymbol{\lambda}' = -\boldsymbol{\lambda}$, then $-T \in \Delta_+(G^{\mathbf{0}}_{\boldsymbol{\lambda}'})$ for each cell $T \in \Delta_+(G^{\mathbf{0}}_{\boldsymbol{\lambda}})$. It directly follows that $\sigma(x_{\boldsymbol{\lambda}})$ and $\sigma(y_{\boldsymbol{\lambda},j})$ are the upward-pointing inner normal vectors of the simplices in $\Delta_+(G^{\mathbf{0}}_{\boldsymbol{\lambda}})$ for all $\boldsymbol{\lambda}$ satisfying $\lambda_1 = -1$. ∎

Theorem 5.5. *Let N be even. The height function $\omega : S_G \to \mathbb{Z}$ given by*

$$\omega(\mathbf{a}) = \begin{cases} 0 & \text{if } \mathbf{a} = \mathbf{0}, \\ 2 & \text{if } \mathbf{a} = \pm e_1, \\ 1 & \text{otherwise} \end{cases} \tag{5.1}$$

induces a regular unimodular triangulation Δ_N of the point configuration S_{C_N}. Specifically,

$$\Delta_N = \bigcup_{\lambda \in \Lambda_N} \Delta_+(G^{\mathbf{0}}_\lambda) \tag{5.2}$$

Proof. Let Δ_N denote the regular subdivision of P_{C_N} induced by ω. For $\lambda \in \Lambda_N$, consider the vectors $x_\lambda, y_{\lambda,j}, \sigma(x_\lambda), \sigma(y_{\lambda,j})$, as defined in Lemma 5.4. First, we focus on x_λ. We have already seen that each vertex of $G^{\mathbf{0}}_\lambda$ except for $-\lambda_1 e_1$ lies on the hyperplane with normal vector x. In fact, it is straightforward to check that

$$\langle x, -\lambda_j(\mathbf{e}_{j-1} - \mathbf{e}_j) + \mathbf{e}_{n+1} = \lambda_j^2 + 1 > 0$$

for all $1 < j \leq n+1$, and that

$$\langle x, \pm\mathbf{e}_1 + 2\mathbf{e}_{n+1}\rangle = \pm 1 + 2 > 0,$$

so x defines a facet of $\Omega_\omega(P_{C_N})$.

Following this same process, one may verify that $y_{\lambda,j}$ defines a facet containing all vertices of $G^{\mathbf{0}}_\lambda$ except $\lambda_j(\mathbf{e}_{j-1} - \mathbf{e}_j)$. By an argument that is symmetric in the first n coordinates, $\sigma(x_\lambda)$ and $\sigma(y_{\lambda,j})$ also defined facets of $\Omega_\omega(P_{C_N})$.

Ranging over all $\lambda \in \Lambda_N$, we have identified a collection of simplices C that are lower facets of $\Omega_\omega(P_{C_N})$. Projecting each C back down to $\mathbb{R}^n$, we get

$$\bigcup_{\lambda\in\Lambda_N} \Delta_+(G^{\mathbf{0}}_\lambda) \subseteq \Delta_N. \tag{5.3}$$

In fact, this set covers P_{C_N} completely: let $a \in P_{C_N}$. Then for some nonzero $c \in \mathbb{R}$, ca is on the boundary of P_{C_N}. Thus, $ca \in G_{\boldsymbol{\lambda}}$ for some $\lambda \in \Lambda_N$, and $a \in G^{\mathbf{0}}_\lambda$. Therefore, $a \in C$ for some cell $C \in \Delta_+(G^{\mathbf{0}}_\lambda)$.

Together, this shows that Δ_N is a triangulations of P_{C_N}, and is the regular triangulation induced by ω. To see that this triangulation is unimodular, recall that all simplices in Δ_N are unimodularly equivalent to the simplex whose nonzero vertices are

$$\mathbf{e}_0 - \mathbf{e}_1, \mathbf{e}_1 - \mathbf{e}_2, \ldots, \mathbf{e}_{\lfloor \frac{n}{2}\rfloor} - \mathbf{e}_{\lfloor \frac{n}{2}\rfloor+1}, -(\mathbf{e}_{\lfloor \frac{n}{2}\rfloor+1} - \mathbf{e}_{\lfloor \frac{n}{2}\rfloor+2}), \ldots, -(\mathbf{e}_{n-1} - \mathbf{e}_n), -\mathbf{e}_n.$$

Placing these vertices as the columns of a matrix, in this order, results in a lower-triangular matrix with determinant ± 1. Thus, the corresponding simplex, and therefore all simplices in Δ_N, are unimodular. This completes the proof. ∎

Remark 5.6. The direct acyclic decomposition scheme developed in [2] is equivalent to the process of computing a regular subdivision of the adjacency polytope induced by certain 0/1 weights. It is shown that for certain graphs, this process will produced regular unimodular triangulations which is desired due to their connection to *primitive decomposition* of a Kuramoto network. Here, however, we can see this is not possible in general. In particular, with the aid of Macaulay2 [11] to test all $2^9 = 512$ possible 0/1 weight orders for P_{C_4}, we verified that only 4 choices of weights produce a triangulation of the polytope, and of these, none are unimodular. So, in then sense of bounding the heights of the lattice points of $\omega(P_{C_N})$ for all even N, using only nonnegative integer heights, to produce a regular unimodular triangulation, the ω given in this subsection is best possible.

5.2. Odd N. For the entirety of this subsection, we assume that N is odd. Recall that in this case, the facets of Q_N consist of all sets of the form

$$F_{j,\lambda} = \operatorname{conv}\{\lambda_1(-\mathbf{e}_1 - \mathbf{e}_2 - \cdots - \mathbf{e}_n), \lambda_2\mathbf{e}_1, \ldots, \widehat{\lambda_j\mathbf{e}_{j+1}}, \ldots, \lambda_N\mathbf{e}_n\}$$

Tracing this back to P_{C_N}, we find that its facets are of the form

$$G_{\boldsymbol{\lambda}} = \operatorname{conv}\{\lambda_1(\mathbf{e}_0 - \mathbf{e}_1), \ldots, \lambda_j(\widehat{\mathbf{e}_{j-1} - \mathbf{e}_j}), \ldots, \lambda_N(\mathbf{e}_n - \mathbf{e}_N) \mid \lambda \in \Lambda_{j,N}\}$$

Set

$$G^{\mathbf{0}}_{\lambda} = \operatorname{conv}\{\lambda_i(\mathbf{e}_i - \mathbf{e}_{i+1}) \mid (\lambda_1, \ldots, \lambda_N) \in \Lambda_{j,N}\},$$

and let

$$\Delta_N = \{G^{\mathbf{0}}_{\lambda} \mid \lambda \in \Lambda_N\}. \tag{5.4}$$

By construction, since each $G_{\boldsymbol{\lambda}}$ is a simplex, Δ_N is a triangulation of P_{C_N}. It is straightforward to check that the matrix whose columns are the nonzero vertices of $G^{\mathbf{0}}_{\lambda}$ has determinant ± 1 for each $\lambda \in \Lambda_N$, so Δ_N is a unimodular triangulation.

Now, let $\omega : S_{C_N} \to \mathbb{Z}$ be the height function given by

$$\omega(\mathbf{a}) = \begin{cases} 0 & \text{if } \mathbf{a} = \mathbf{0} \\ 1 & \text{otherwise.} \end{cases} \tag{5.5}$$

It is clear from this choice that the lower facets of the lifted polytope $\Omega_\omega(P_{C_N})$ are of the form

$$\operatorname{conv}\{\mathbf{0}, G_{\boldsymbol{\lambda}} \times \{1\}\},$$

so their projections back onto $\mathbb{R}^n$ are exactly the simplices $G^{\mathbf{0}}_{\lambda}$ for all $\lambda \in \Lambda_N$. With this work, we have shown the following.

Proposition 5.7. *The set Δ_N is a regular, unimodular triangulation of P_{C_N}, and is induced by the height function ω in (5.5).*

We can, in fact, be more specific when identifying the lower facets of $\Omega_\omega(P_{C_N})$.

Corollary 5.8. *The upward-pointing inner normal vectors for $\Omega_\omega(G^{\mathbf{0}}_{\lambda})$ are $x = (x_1, \ldots, x_N)$ where*

$$x_k = \begin{cases} \sum_{i=1}^{k} \lambda_i & \text{if } i < N, \\ 1 & \text{if } i = N \end{cases}$$

for all $\lambda \in \Lambda_N$.

Proof. Let $\Omega_\omega(G^{\mathbf{0}}_{\boldsymbol{\lambda}})$ be a lower facet of $\Omega_\omega(P_{C_N})$ for some $\boldsymbol{\lambda} \in \Lambda_{j,N}$, and select a nonzero vertex $\mathbf{v}$ of the facet. Since this vertex is nonzero, we know $\mathbf{v}$ is of the form $\mathbf{v} = \lambda_{r+1}(\mathbf{e}_r - \mathbf{e}_{r+1} + \mathbf{e}_N$ for some $r \neq j$. Then

$$\langle \mathbf{x}, \mathbf{v} \rangle = \lambda_{r+1} \left(\sum_{i=1}^{r} \lambda_i - \sum_{l=1}^{r+1} \lambda_l \right) + 1 = -\lambda_{r+1}^2 + 1 = 0.$$

Thus, $\mathbf{x}$ is normal to $\Omega_\omega(G^{\mathbf{0}}_{\boldsymbol{\lambda}})$. ∎

6. Cell enumeration algorithm. In this section, we briefly summarize the algorithm for for constructing a regular unimodular triangulation for the adjacency polytope P_{C_N} as proposed above. Here, we shall focus only on the enumeration of all the upward pointing inner normal vectors of the lifted polytope $\Omega(P_{C_N})$ of the point configuration S_{C_N}, since these are directed used in the homotopy construction to be described in section 8. Moreover, these

objects directly correspond to tropical stable intersections as we shall discuss in detail in subsection 9.2. Once a normal vector $\mathbf{v}$ is obtained, the vertices of the corresponding cell can be found easily by computing the minimizing set of the linear functional $\langle \cdot\,, \mathbf{v} \rangle$.

The algorithm EnumerateNormals(N) for enumerating inner normal vectors is listed in Algorithm 6.1. It takes the argument N, which is the number of nodes in the cycle graph and produces the set of all upward pointing inner normal vectors of the lower facets of $\Omega(P_{C_N})$ which are in one-to-one correspondence with the simplices in the regular unimodular triangulation Δ_N. It is important to note that this algorithm is *pleasantly parallel* since the description of vectors associated with indices $\boldsymbol{\lambda} \in \Lambda_N$ are independent from one another.

Algorithm 6.1 EnumerateNormals (N): Enumeration of upward pointing inner normals

Input: $N \in \mathbb{Z}^+, N > 2$.
Output: Set C of all upward pointing inner normals.

```
C ← ∅
for all (λ_1, ..., λ_n) ∈ Λ_N do
  for k = 1, ..., n do
    x_k ← Σ_{i=1}^{k} λ_i
  end for
  x_N ← 1
  x ← [x_1, ..., x_N]
  C ← C ∪ {x}
  if N is even and N > 2 then
    for j = 1, ..., n do
      if λ_j = 1 then
        y ← x + Σ_{k=1}^{j-1} e_k
        C ← C ∪ {y}
      end if
    end for
  end if
end for
return C
```

7. Software implementation. The main algorithms for generating the cells in the regular unimodular triangulation Δ_N of P_{C_N} is implemented in an open source Python package called `kap-cycle` [4] which is freely available at https://github.com/chentianran/kap-cycle. In addition to the cells, this package also produces the upward pointing inner normals corresponding to each cell. That is, it provides all the necessary information for bootstrapping the adjacency polytope homotopy proposed in section 8.

8. The Adjacency Polytope Homotopy for Kuramoto Equations. We now return to the problem of find all isolated complex solutions of (3.1). Equivalently, these are the solutions of $F^R_{C_N}$ defined in (3.2). Utilizing the unimodular regular triangulation of the adjacency polytope P_{C_N}, in this section we describe a specialized *polyhedral homotopy* [12] construction for locating all of these complex solutions.

Consider the function $H_{C_N} : \mathbb{C}^n \times \mathbb{C} \to \mathbb{C}^n$ with $H_{C_N}(\mathbf{x}, t) = (H_{C_N,1}, \ldots, H_{C_N,n})$ given by

$$H_{C_N,k}(x_1, \ldots, x_n, t) = c_k^R - \sum_{\{i,j\} \in \mathcal{E}(C_N)} a_{ijk}^R \left(\frac{x_i}{x_j} - \frac{x_j}{x_i} \right) t^{\omega_{ij}} \quad \text{for } k = 1, \ldots, n \tag{8.1}$$

where $\omega_{ij} = \omega(\mathbf{e}_i - \mathbf{e}_j)$ as given in (5.1) or (5.5) depending on the parity of N. Clearly, $H_{C_N}(\mathbf{x}, 1) = F_{C_N}^R(\mathbf{x})$. As t varies strictly between 0 and 1 within the interval $[0, 1]$, $H_{C_N}(\mathbf{x}, t)$ represents a smooth deformation of the system $F_{C_N}^R$. We shall show that under this deformation, the corresponding complex roots also vary smoothly. Thus, the deformation forms smooth paths reaching the complex roots of $F_{C_N}^R$ and, equivalently, that of the algebraic synchronization system F_{C_N}.

Proposition 8.1. *For generic choices of $\{c_i\}_{i=1}^n$ and $\{a'_{ij}\}_{\{i,j\} \in \mathcal{E}(C_N)}$, the solution set of $H_{C_N}(\mathbf{x}, t) = \mathbf{0}$ within $\mathbb{C}^n \times (0, 1)$ consists of $N\binom{N-1}{\lfloor (N-1)/2 \rfloor}$ smooth curves that are smoothly parametrized by $t \in (0, 1)$, and the limit point of these curves as $t \to 1$ are precisely the isolated complex solutions of $F_{C_N}(\mathbf{x}) = \mathbf{0}$.*

Proof. As proved in [5], for generic choices of $\{c_i\}_{i=1}^n$ and $\{a'_{ij}\}_{\{i,j\} \in \mathcal{E}(C_N)}$, the system $H_{C_N}(\mathbf{x}, 1) \equiv F_{C_N}^R$ is in general position with respect to the adjacency polytope bound, i.e., it has the maximum number of isolated complex solutions. Therefore, without loss of generality, we can assume the choices of a_{ijk}^R and c_k^R for $k = 1, \ldots, n$ and $\{i, j\} \in \mathcal{E}(C_N)$ in (8.1) are generic, that is, $H(\mathbf{x}, 1)$ is in general position with respect to the adjacency polytope bound.

Also note that for any $t \neq 0$, $H(\mathbf{x}, t)$ is has the same form as (3.2) since the effect of t is only scaling the coefficients. We shall show that $H(\mathbf{x}, t)$ remains a generic member of (3.1) for all $t \in (0, 1]$ and hence the all complex solutions of $H(\mathbf{x}, t) = \mathbf{0}$ (as a system in $\mathbf{x}$ only) are isolated and the total number matches the adjacency polytope bound $N\binom{N-1}{\lfloor (N-1)/2 \rfloor}$.

As noted in Remark 4.1, the genericity condition is characterized by an algebraic function D, the discriminant, which is a polynomial in the coefficients $\{c_i^R\}_{i=1}^n$ and $\{a_{ij}^R t^{\omega_{ij}}\}_{\{i,j\} \in \mathcal{E}(C_N)}$, and $F(\mathbf{x}) = H_{C_N}(\mathbf{x}, t)$ is generic with respect to the adjacent polytope bound precisely when $D \neq 0$. Consider the univariate polynomial

$$g(t) = D((c_i^R)_{i=1}^n, (a_{ij}^R t^{\omega_{ij}})_{\{i,j\} \in \mathcal{E}(C_N)}).$$

By our genericity assumption, $g(1) \neq 0$, and therefore the polynomial $g(t)$ cannot be the zero polynomial. It then has finitely many zeros within the unit disk of $\mathbb{C}$, say, $r_1 e^{\tau_1}, \ldots, r_\ell e^{\tau_\ell}$ for some $\ell \in \mathbb{Z}^+$.

Picking a real value $\tau \in [0, 2\pi]$ such that $\tau \neq \tau_k$ for $k = 1, \ldots, \ell$ will ensure $g(e^\tau t) \neq 0$ for all $t \in (0, 1)$. But $g(e^\tau t)$ describes the discriminant condition for the system

$$H_{C_N,i}(x_1, \ldots, x_n, e^\tau t) = c_i^R - \sum_{\{i,j\} \in \mathcal{E}(C_N)} (a_{ij}^R e^{\omega_{ij}\tau}) \left(\frac{x_i}{x_j} - \frac{x_j}{x_i} \right) t^{\omega_{ij}},$$

which implies $H_{C_N,i}(\mathbf{x}, e^\tau t)$ is in general position for all $t \in (0, 1)$.

Notice that the map $\tau \mapsto e^{\omega_{ij}\tau}$ is finite-to-one, and the map $a_{ij}^R \mapsto e^{\omega_{ij}\tau}$ is a nonsingular linear transformations on the coefficients a_{ij}^R which preserves genericity. Thus we can conclude

that for generic choices of a_{ij}^R and c_i^R, $H_{C_N}(\mathbf{x}, t)$ will be in general position over the entire t-interval of $(0, 1)$.

This shows that at any fixed $t \in (0, 1)$, all solutions of $H(\mathbf{x}, t) = \mathbf{0}$ in $\mathbb{C}^n$ are isolated and the total number is exactly the adjacency polytope bound $N\binom{N-1}{\lfloor (N-1)/2 \rfloor}$. A direct application of the homotopy continuation theory is then sufficient to establish that the solution set of $H(\mathbf{x}, t) = \mathbf{0}$ in $\mathbb{C}^n \times (0, 1)$ forms paths that are smoothly parametrized by t. Furthermore, by continuity, the limit points of these paths as $t \to 1$ must be all the solutions of $F_{C_N}^R(\mathbf{x}) = \mathbf{0}$ which is identical to that of $F_{C_N}(\mathbf{x}) = \mathbf{0}$. ∎

The equation $H_{C_N}(\mathbf{x}, t) = \mathbf{0}$ defines finitely many smooth paths in $\mathbb{C}^n \times (0, 1)$ reaching all of the isolated complex solutions of the target synchronization system $F_{C_N}(\mathbf{x}) = \mathbf{0}$. The starting point of these paths at $t = 0$, however, cannot be determined directly since $H_{C_N}(\mathbf{x}, 0) = (c_1, \ldots, c_n)$ which has no root in $\mathbb{C}^n$. This obstacle is surmounted via a technique similar to the main construction in polyhedral homotopy [12].

Recall that Δ_N is the set of cells forming the unimodular triangulation of the adjacency polytope P_{C_N} (defined in (5.2) or (5.2) depending on the parity of N). For each cell $T \in \Delta_N$, we define the subset of (directed) edges

$$\mathcal{E}(T) = \{(i, j) \in \mathcal{E}(G) \mid \mathbf{e}_i - \mathbf{e}_j \in T\}.$$

Here, we do not assume the symmetry of edges, i.e., $(i, j) \in \mathcal{E}(T)$ does not imply $(j, i) \in \mathcal{E}(T)$. Define the *cell system* $F_T = (F_{T,1}, \ldots, F_{T,n})$ associated with the cell $T \in \Delta_N$ given by

$$F_{T,k}(\mathbf{x}) = c_k^R - \sum_{(i,j)\in\mathcal{E}(T)} a_{ijk}^R \frac{x_i}{x_j} \quad \text{for } k = 1, \ldots, n. \tag{8.2}$$

This system can be considered as a subsystem of the unmixed synchronization system (3.2) in the sense that it involves a subset of the terms in that system: only those terms corresponding to points in T. Indeed, T is exactly the Newton polytope of the corresponding cell system.

Remark 8.2. The cell systems defined here are refinements of the *facet systems* studied in [2]. Indeed, for odd N, they are exactly the facet systems since each $T \in \Delta_N$ is the convex hull of a facet of the adjacency polytope P_{C_N} together with the origin. For even N values, however, the cell systems will be significant refinement of facet systems defined in [2]. This distinction will be explained in detail in subsection 9.1.

Here, each cell $T \in \Delta_N$ is a full-dimensional lattice simplex with normalized volume 1 (a primitive simplex). From classical theory from toric algebraic geometry, we can deduce that the corresponding cell system has a unique solution.

Lemma 8.3. *For generic choices of $\{c_i\}_{i=1}^n$ and $\{a'_{ij}\}_{\{i,j\}\in\mathcal{E}(C_N)}$, each system of Laurent polynomial equations $F_T(\mathbf{x}) = \mathbf{0}$ for $T \in \Delta_N$ has a unique complex solution, and this solution is isolated and nonsingular.*

With this result, we construct a modification of the of the homotopy (8.1) that will define a solution path that starts from the unique solution of the cell system F_T. This is essentially a specialized polyhedral homotopy [12] construction using the triangulation found in the previous section.

Definition 8.4 (Adjacency polytope homotopy). *For each cell $T \in \Delta_N$, let $(\alpha_1, \ldots, \alpha_n, 1) \in \mathbb{R}^{n+1}$ be the associated upward pointing inner normal vector as given in Lemma 5.4 and Corollary 5.8. We define the* adjacency polytope homotopy *induced by this cell as the function $H_T = (H_{T,1}, \ldots, H_{T,n}) : \mathbb{C}^n \times [0,1]$ given by*

$$H_{T,k}(\mathbf{y}, t) = H_{C_N,k}(y_1\, t^{\alpha_1}\, , \ \ldots, \ y_n\, t^{\alpha_n}). \tag{8.3}$$

Recall that the collection of cells in Δ_N form a regular triangulation of the adjacency polytope P_{C_N} which is also the Newton polytope of (3.2) By applying the construction of unmixed form of polyhedral homotopy [12], we obtain the desired result: A homotopy construction that can be used to locate all isolated complex solution of the algebraic synchronization equation (3.1).

Theorem 8.5. *For generic choices of $\{c_i\}_{i=1}^n$ and $\{a'_{ij}\}_{\{i,j\}\in\mathcal{E}(C_N)}$*

1. *The solution set of $H_T = \mathbf{0}$ within $\mathbb{C}^n \times (0,1)$ consists of a finite number of smooth paths parametrized by t, and the limit point of these paths as $t \to 1$ are precisely the isolated solutions of $F_{C_N}(\mathbf{x}) = \mathbf{0}$ in $\mathbb{C}^n$.*
2. *Among them, there is a unique path $C_T(t)$ whose limit point $C_T(0) = \lim_{t\to 0^+} C_T(t)$ is the unique solution of $F_T(\mathbf{x}) = \mathbf{0}$.*
3. *The set of end points $\{C_T(1) \mid T \in \Delta_N\}$ of paths induced by all cells is exactly the isolated $\mathbb{C}$-solution set of $F_{C_N}(\mathbf{x}) = \mathbf{0}$.*

Remark 8.6. From the viewpoint of numerical analysis, the stability of a homotopy formulation is a deep and complex problem that is outside the scope of this paper. Here we only comment one distinct advantage of the adjacency polytope homotopy over a direct application of polyhedral homotopy method.

In practical implementations of polyhedral homotopy, it is well known that the distribution of the exponents of the t parameter in the homotopy plays a crucial role in the numerical stability of the homotopy algorithm [10]. In particular, if the exponents of t spread over a wide range, the problem of tracking the homotopy paths can become extremely ill-conditioned and standard algorithms for path tracking become unstable. While many techniques have been developed to deal with this issue. It is clearly best if this problem can be avoided in the first place. In our construction, the exponents of t in both (8.1) and Definition 8.4 involve small integers, this ensures that the exponents of t in $H_T(\mathbf{y}, t)$ consist of only small positive integers for relatively small N values.

9. Interpretations.

In this section we interpret our main results in a wider context and draw connections to closely related problems. Even though the main goal is to construct an efficient homotopy method for locating complex synchronization configurations for Kuramoto networks supported on cycle graphs, we shall show that our construction actually provides explicit solutions to other problems: the direct acyclic decomposition of cycle Kuramoto networks, and the self-intersection of a tropical hypersurface.

9.1. Direct acyclic decomposition of cycle networks.

In the recent work [2], a general scheme is proposed to decompose a Kuramoto network into smaller subnetworks supported by direct acyclic graphs while preserve certain properties of the synchronization configurations.

This scheme utilize the geometric properties of the adjacent polytope. Indeed, the subnetworks are in one-to-one correspondence with the facets of the adjacency polytope.

In this context, the constructions proposed in this paper provide two important improvement to that decomposition scheme. First, the regular unimodular triangulation of the adjacency polytope P_{C_N} gives rise to a significant refinement for the direct acyclic decomposition scheme which will decompose a cycle network into "primitive" subnetworks. This was not possible for even N values with the original decomposition scheme. Second, as the starting system induced by the adjacency polytope homotopy (8.1) can be reduced into binomial systems and hence can be solved easily and efficiently. Finally, since the explicit formula for the generic root count is known (section 4), the number of solution paths induced by the homotopy proposed in section 8 matches the generic root count exactly, and thus will not produce extraneous solution paths in the generic situation. This feature was not established in the original decomposition scheme and the associated homotopy construction.

To see the regular unimodular triangulation described in section 5 give rises to a decomposition of the Kuramoto network, we first define the subnetwork corresponds to a cell.

Definition 9.1 (Directed acyclic subnetwork). *Let Δ_N be the regular unimodular triangulation of P_{C_N} defined in (5.2) and (5.4). For each cell $T \in \Delta_N$, we define the* directed acyclic subnetwork *associated with T to be the graph $(\{0, \ldots, N-1\}, \mathcal{E}(T))$ where*

$$\mathcal{E}(T) = \{(i, j) \in \mathcal{E}(C_N) \mid \mathbf{e}_i - \mathbf{e}_j \in T\}.$$

This is a refinement of the definition given in [2] where subnetworks correspond to facets of the adjacency polytope. In contrast, subnetworks defined above come from a triangulation which, in the case of even N values, are associated with simplices in the facets of P_{C_N}.

As established in [2], such a subnetwork associated with a cell is always an *acyclic* graph which justifies its name (directed acyclic subnetwork). Moreover, such subnetworks are of the simplest possible form known as "primitive" subnetworks.

Definition 9.2 (Primitive directed acyclic subnetwork). *A subnetwork associated with a cell, as defined above, is said to be* primitive *if it contains exactly $n = N - 1$ directed edges.*

Note that since each cell $T \in \Delta_N$ is a simplex of dimension n and contains exactly n nonzero points of the form $\mathbf{e}_i - \mathbf{e}_j$ for $i \neq j$, we can see the induced subnetwork must be primitive.

Proposition 9.3. *Let Δ_N be the regular unimodular triangulation of P_{C_N}. For each cell $T \in \Delta_N$, the associated directed acyclic subnetwork is primitive.*

Figure 2 shows the direct acyclic subnetworks of a cycle network with 4 nodes induced by the triangulation Δ_N. Clearly, all subnetworks are primitive. In contrast, the original decomposition scheme for the same network, as shown in Figure 3, produces subnetworks that are not primitive.

9.2. A tropical interpretation. Even though it was not stated explicitly, the procedure that resulted in the adjacency polytope homotopy (8.1) is actually rooted from tropical algebraic geometry [17]. In this section, we provide the interpretation from the tropical viewpoint.

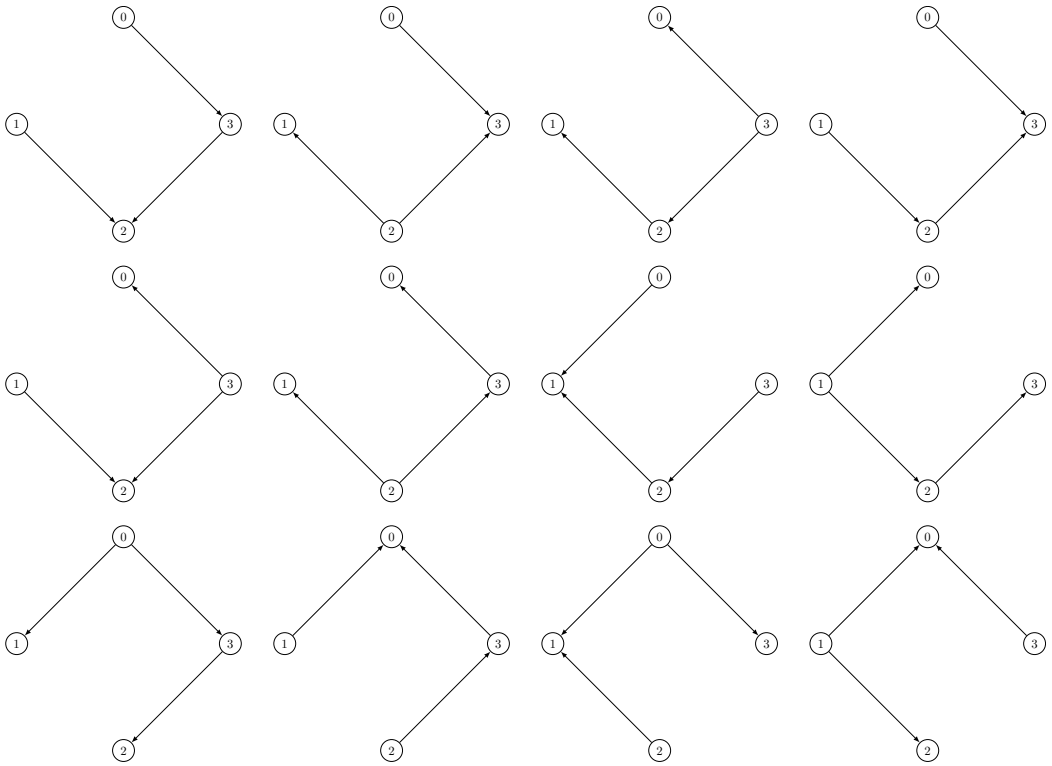

Figure 2. *Direct acyclic subnetworks of a cycle network with 4 nodes induced by the most refined decomposition scheme developed in this paper. Every subnetwork is* primitive. *This is to be compared with the original coarser decomposition scheme shown in Figure 3*

Recall that we started with the unmixed form of the algebraic synchronization equation (3.2). If we consider the valuation on the field of coefficients given by

$$\text{val}(c_i^R) = 0 \qquad \text{and} \qquad \text{val}(a_{ijk}^R) = \begin{cases} 2 & \text{if } N \text{ is even and } (i,j) = (1,0) \\ 1 & \text{otherwise,} \end{cases} \tag{9.1}$$

which mirrors the choices of the weights given in (5.1) and (5.5), then the tropicalization of the n polynomials in (1.1) are identical and they define a common tropical hypersurface. The main results developed in section 5 can thus be interpreted tropically: The valuation defined above induces the simplest (stable) self-intersection.

Proposition 9.4. *Let $h = \text{trop}(F^R_{C_N,k})$ for $k = 1, \ldots, n$ be the tropicalization of (3.2) with respect to the valuation given in (9.1). Then the tropical hypersurface defined by h has exactly $N\binom{N-1}{\lfloor (N-1)/2 \rfloor}$ self-intersection points, and each intersection is of multiplicity one.*

As discussed in Remark 5.6, the special choice of the valuation (9.1) is an important condition for this result to hold. Using only 0-1 valuations, for example, will not produce self-intersections with multiplicity one. One of the key contribution of this paper is the explicit formula for these self-intersection points. These tropical self-intersection points are precisely the tropicalizations of the curves defined by (8.1).

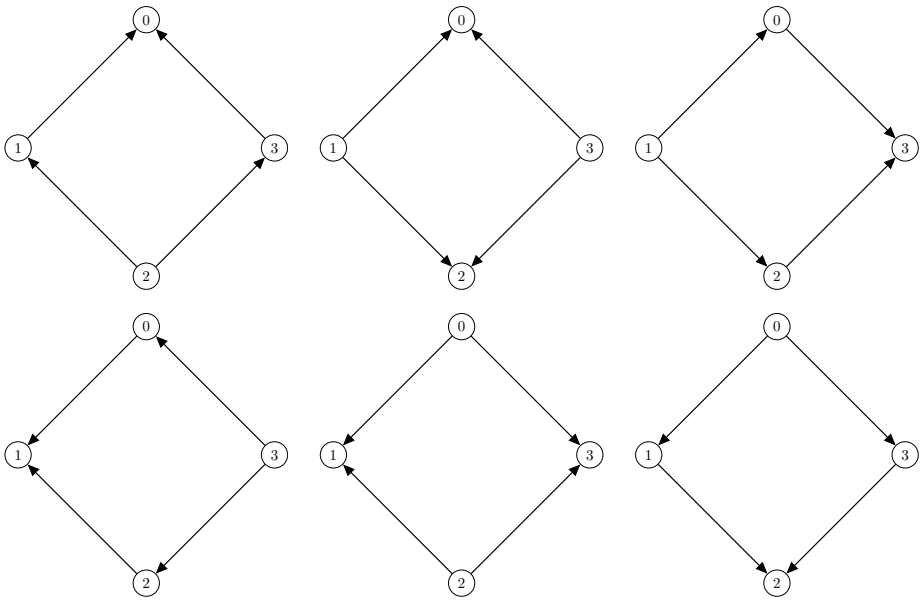

Figure 3. *Direct acyclic subnetworks of a cycle network with 4 nodes induced by the coarser decomposition scheme originally proposed in [2]. None of the subnetworks are primitive.*

9.3. Equivalence of interpretations. With the above interpretations, we have the equivalence of three important class of problems from different context, as shown in Table 1. This connection, especially the connection between regular subdivision of adjacency polytope and direct acyclic decomposition was first proposed in [2]. In this paper, we further refine this idea and provided explicit answers for problems in the bottom row of Table 1 in the cases of cycle networks.

Adjacency polytope	Kuramoto network	Tropical hypersurface
Regular subdivision	Directed acyclic decomposition	Stable self-intersections
Regular unimodular triangulation	Directed acyclic decomposition into primitive subnetworks	Stable self-intersections with multiplicity one

Table 1
The 3-way dictionary that translates equivalent concepts among the three different points of view.

10. Conclusions. Following the volume computation result in [5], this paper aims to deepen the geometric understanding of adjacency polytopes associated to a cycle Kuramoto network and use these geometric information to explore three different aspects of Kuramoto equations:

1. To create an efficient polyhedral-like homotopy algorithm for solving Kuramoto equations;
2. To explicitly describe direct acyclic decompositions of Kuramoto networks into

primitive subnetworks; and

3. To understand the stable intersections of the tropical hypersurfaces defined by Kuramoto equations.

First, we derived the explicit formula for a regular unimodular triangulation of the adjacency polytope P_{C_N} associated to a cycle graph of N nodes for any $N > 2$. This greatly strengthens the results from [5] where only the normalized volume of P_{C_N} is known.

Then, using this regular unimodular triangulation, we develop a homotopy continuation algorithm that is similar to the well established polyhedral homotopy method but has the distinct advantage that it completely sidesteps the costly mixed volume/cells computation step. This homotopy is also a significant improvement over the direct acyclic homotopy proposed in [2] since this homotopy deforms the Kuramoto system into simplest possible subsystems each having a unique solution. From the computational viewpoint, the proposed homotopy also offers important advantages in numerical conditions, efficiency, and scalability as discussed in Remark 8.6.

The third contribution of this paper is a significantly refined version of the direct acyclic decomposition scheme originally proposed in [2]. The regular unimodular triangulation proposed here induces a decomposition of a cycle Kuramoto network into the smallest possible components known as primitive subnetworks. Primitive subnetworks are of great value since they each have a unique complex synchronization configuration which can be computed easily and efficiently. This is to be compared with the situation of the original decomposition scheme where the resulting subnetworks, in general, may not be primitive.

Finally, we showed the significance of our results in the context of tropical geometry. In particular, we produced explicit formula for all stable intersections of the tropical hypersurfaces defined by the unmixed form of the Kuramoto system under a special choice of the valuation. This valuation induces particularly nice tropical intersections as we shown that every intersection point is of multiplicity 1.

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