GRAPH EDGE CONTRACTION AND ADJACENCY POLYTOPES

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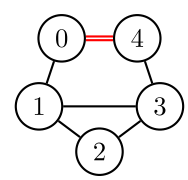
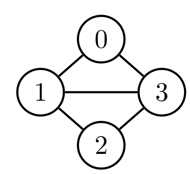
Abstract. Adjacency polytopes, a.k.a. symmetric edge polytopes, associated with undirected graphs have been proposed and studied in several seemingly independent areas ranging from number theory to discrete geometry and the study of Kuramoto models. Regular subdivisions of adjacency polytopes are of particular importance in solving certain algebraic systems of equations. This paper explores the connection between the regular subdivisions of an adjacency polytope and the contraction of the underlying graph along an edge. The main result is the construction of a special regular subdivision whose cells are in one-to-one correspondence with facets of adjacency polytope associated with an edge-contraction of the original graph.

1. Introduction

For a connected graph G with vertices $\mathcal{V}(G) = \{0, 1, \ldots, n\}$ and edge set $\mathcal{E}(G)$, its *adjacency polytope* [2] (a.k.a. *symmetric edge polytope* [12]) is the convex polytope $\nabla_G = \operatorname{conv}\{\mathbf{e}_i - \mathbf{e}_j \mid \{i, j\} \in \mathcal{E}(G)\}$. In the context of Kuramoto models [10], the geometric structure of adjacency polytopes turned out to be instrumental in understanding the root counting problem of algebraic Kuramoto equations [3, 4, 10]. In the broader context, the adjacency polytope of a graph is the symmetric edge polytope which has been studied by number theorists, combinatorialists, and discrete geometers motivated by several seemingly independent problems [6, 8, 9, 12, 13, 14, 15]. These different viewpoints are consolidated in the recent work by D'Alì, Delucchi, and Michałek [5], which, among other contributions, shed new light on the structure of adjacency polytopes associated with graphs consisting of two subgraphs sharing a single edge. In particular, using Gröbner bases methods, the authors provided explicit formulae for the number of facets and the normalized volume of adjacency polytopes associated with graphs formed by gluing together two connected bipartite graphs, trees, or



(A) Graph G



(B) $G' = G /\!\!/ \{0, 4\}$

Figure 1. Edge contraction of a graph

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cycles. In this paper, we pursue this line of inquiry by considering the effect of a contraction of a graph along an edge on the corresponding adjacency polytope.

Of particular importance in the study of algebraic Kuramoto equations derived from a graph G are the facets and regular subdivisions of the corresponding adjacency polytope ∇_G. The set of facets of ∇_G corresponds to directed acyclic subgraphs of G that satisfy certain minimal flow property [1]. Geometric structure of the facets played a key role in computing the volume of the adjacency polytope for certain families of graphs [3, 4, 5] as well as determining the generic root count for algebraic Kuramoto equations. Equally important, a nontrivial regular subdivision of ∇_G gave rise to a toric deformation of the underlying algebraic Kuramoto equations into a simpler system of equations, solutions to which can be identified with all the complex solutions to the original system. In this paper, we explore the connection between the regular subdivision of ∇_G and the facets of $\nabla_{G /\!\!/ e}$ where $G /\!\!/ e$ is the contraction of G along an edge $e \in \mathcal{E}(G)$. See example in Figure 1.

The main contribution of this paper is the construction of a special regular subdivision of ∇_G whose cells are in one-to-one correspondence with facets of $\nabla_{G /\!\!/ e}$ associated with the graph $G /\!\!/ e$. We also show that if G consists of two subgraphs G_1 and G_2 sharing exactly one edge e, then the cells in the special regular subdivision of ∇_G are in one-to-one correspondence with the products of facets of the adjacency polytopes $\nabla_{G_1'}$ and $\nabla_{G_2'}$, where $G_1' = G_1 /\!\!/ e$ and $G_2' = G_2 /\!\!/ e$. Since this study is largely motivated by the tropical intersections problem derived from the algebraic Kuramoto equations, the resulting subdivision of the polytope ∇_G can be also conveniently viewed as a subdivision of the underlying point configuration. That is, every cell in the subdivision that we aim to produce is a convex hull of vertices of ∇_G which are all integer lattice points. Combined with the existing knowledge of facets of adjacency polytopes (symmetric edge polytopes) associated with trees, cycles, bipartite graphs, and wheel graphs [5], this result enables us to study more complicated graphs formed by gluing these basic building blocks along the edges.

This paper is structured as follows. Section 2 reviews the necessary definitions and notations. Section 3 defines the special regular subdivision induced by an edge contraction and establishes the main results. We conclude with an interpretation of the results in Section 4.

2. Preliminaries and notations

Given a graph $G = (V, E)$ and an edge $e = \{a, b\} \in E$, the *contraction* $G /\!\!/ e$ of G along e is a new graph obtained by merging the two nodes a and b in G. That is, $G /\!\!/ e = (V', E')$ with $V' = V \setminus \{a, b\} \cup \{a'\}$ and $E' = E \setminus \{a, b\} \cup \{\{a', v\} \mid v \neq a, b, \{a, v\} \in E \text{ or } \{b, v\} \in E\}$.

A *convex polytope* is the convex null of a finite set of points. Its *dimension* is the dimension of the smallest affine space that contains it. A (nonempty) *face* of a convex polytope is a subset of the polytope on which a linear functional $\langle \cdot , \boldsymbol{\alpha} \rangle$ is minimized. In this case, $\boldsymbol{\alpha}$ is an inner normal vector of the face. Faces are themselves polytopes, and proper faces of the maximal dimension are called *facets*. In this paper, we only deal with (convex) *lattice polytopes*, i.e., the convex polytopes whose vertices have integer coordinates. For an n-dimensional lattice polytope $P \subset \mathbb{R}^n$, its *normalized volume*, denoted by $\operatorname{nvol}(P)$, is $n! \operatorname{vol}(P)$, which is always an integer.

For a connected graph G with vertices $\mathcal{V}(G) = \{0, 1, \ldots, n\}$ and (undirected) edge set $\mathcal{E}(G)$, its *adjacency polytope* [2] (a.k.a. *symmetric edge polytope* [5, 12]) is the convex polytope

$$\nabla_G = \operatorname{conv}\{\mathbf{e}_i - \mathbf{e}_j \mid \{i, j\} \in \mathcal{E}(G)\} \subset \mathbb{R}^n, \tag{1}$$

where $\mathbf{e}_i \in \mathbb{R}^n$ is the vector with 1 in the i-th entry and zero elsewhere, and $\mathbf{e}_0 = \mathbf{0}$.

In the trivial case of $n = 0$, ∇_G is simply $\{\mathbf{0}\}$, and we adopt the convention that the only facet of ∇_G is $\varnothing$. For $n > 0$, ∇_G is a full-dimensional polytope in $\mathbb{R}^n$. We also extend this construction to subgraphs of G. For a subgraph G' of G, we use the identical definition $\nabla_{G'} = \operatorname{conv}\{\mathbf{e}_i - \mathbf{e}_j \mid \{i, j\} \in \mathcal{E}(G')\}$. In this case, however, $\nabla_{G'} \subset \operatorname{span}\{\mathbf{e}_i \mid i \in \mathcal{V}(G')\}$ and may not be full-dimensional. The set of facets of ∇_G is denoted by $\mathcal{F}(\nabla_G)$. Any facet of ∇_G is an intersection of this polytope with a supporting hyperplane, which is uniquely determined by an inner normal vector [16]. Moreover, by construction, $\mathbf{0}$ is an interior point of ∇_G for $n > 0$, which allows the inner normal vectors to be normalized to a certain form. We state this observation as a lemma for later reference.

Lemma 1. *For any graph G such that ∇_G is full-dimensional in $\mathbb{R}^n$, a nonzero vector $\boldsymbol{\alpha}$ defines a facet of ∇_G if and only if there are $x_1, \ldots, x_n$ such that $x_1, \ldots, x_n$ are linearly independent as vectors and*

$$\begin{aligned} \langle \mathbf{x}_i , \boldsymbol{\alpha} \rangle &= -1 \quad \text{for any } i = 1, \ldots, n, \text{ and} \\ \langle \mathbf{x} , \boldsymbol{\alpha} \rangle &\geq -1 \quad \text{for any } \mathbf{x} \in \nabla_G. \end{aligned}$$

If ∇_G is not full-dimensional in $\mathbb{R}^n$, the above statement remains valid in the coordinate subspace in which ∇_G is full-dimensional.

A (polyhedral) subdivision of a convex polytope P is a collection $\mathcal{D}$ of convex polytopes contained in P and of the same dimension as P such that their union is P and the intersection of any two is their (possibly empty) common face. Elements of a subdivisions are known as cells. A *point configuration* is a finite collection of labeled points $S \subset \mathbb{R}^n$ [11]. A subdivision of S is simply a subdivision of $\operatorname{conv}(S)$ whose cells are convex hulls of the subsets of S. For such a cell C, we use the notations $\dim(C) := \dim(\operatorname{conv}(C))$, $\operatorname{vol}(C) := \operatorname{vol}(\operatorname{conv}(C))$ and $\operatorname{nvol}(C) := \operatorname{nvol}(\operatorname{conv}(C))$.

Regular subdivision is a particularly important class of subdivisions. For a point configuration S, using weights assigned by a function $\omega : S \to \mathbb{R}$, we define $\hat{S} = \{(\mathbf{x}, \omega(\mathbf{x})) \mid \mathbf{x} \in S\}$. An inner normal vector $\hat{\boldsymbol{\alpha}} \in \mathbb{R}^{n+1}$ of a face of $\operatorname{conv}(\hat{S})$ is said to be *upward pointing* if $\langle \mathbf{e}_{n+1} , \hat{\boldsymbol{\alpha}} \rangle > 0$. A facet of $\operatorname{conv}(\hat{S})$ with an upward pointing inner normal vector is called a *lower facet*. The projection of all lower facets of $\operatorname{conv} \hat{S}$ form a subdivision of S, the *regular subdivision* (a.k.a. coherent subdivision) of S induced by weight function ω [7, 11]. In this case, a lower facet is defined by a vector $\boldsymbol{\alpha} \in \mathbb{R}^n$, a value $h \in \mathbb{R}$ and a set $C \subset S$ with $|C| \geq n$, $\dim(\operatorname{conv}(C)) = n$ such that

$$\begin{aligned} \langle \mathbf{x} , \boldsymbol{\alpha} \rangle + \omega(\mathbf{x}) &= h \quad \text{for all } \mathbf{x} \in C, \\ \langle \mathbf{x} , \boldsymbol{\alpha} \rangle + \omega(\mathbf{x}) &> h \quad \text{for all } \mathbf{x} \in S \setminus C. \end{aligned} \tag{2}$$

The construction of a special regular subdivision of an adjacency polytope induced by an edge contraction of the underlying graph is the main focus of this paper.

3. Regular subdivision induced by edge contraction

As noted in the definition of adjacency polytopes, we identify edges of a graph of $n+1$ nodes with points in $\mathbb{R}^n$ via the map

$$\phi(i,j) = \mathbf{e}_i - \mathbf{e}_j. \tag{3}$$

Here, an undirected edge $\{i,j\} \in \mathcal{E}(G)$ is considered as a pair of directed edges (i,j) and (j,i). With this, the adjacency polytope of G is simply $\operatorname{conv}(\phi(\mathcal{E}(G)))$. In this section, we construct a regular subdivision of ∇_G induced by an edge contraction.

Definition 1 (Edge contraction subdivision)**.** For an edge $\{k_1,k_2\} \in \mathcal{E}(G)$ to be contracted, we define the lifting function $\omega_{k_1,k_2} : \phi(\mathcal{E}(G)) \to \mathbb{Z}$ given by

$$\omega_{k_1,k_2}(\mathbf{e}_i - \mathbf{e}_j) = \begin{cases} 0 & \text{if } \{i,j\} = \{0,k\}, \\ 1 & \text{otherise,} \end{cases} \tag{4}$$

and the resulting lifted polytope

$$\hat{\nabla}_G = \operatorname{conv}\{(\mathbf{e}_i - \mathbf{e}_j, \omega_{k_1,k_2}(\mathbf{e}_i - \mathbf{e}_j)) \mid \{i,j\} \in \mathcal{E}(G)\} \subset \mathbb{R}^{n+1}. \tag{5}$$

The projections of the facets on the lower hull of $\hat{\nabla}_G$ onto $\mathbb{R}^n \times \{0\}$ form a subdivision of ∇_G, the regular subdivision induced by ω_{k_1,k_2}. This subdivision, denoted by $\mathcal{D}_{k_1,k_2}$ will be referred to as the *edge contraction subdivision* of ∇_G induced by the edge contraction of G along $\{k_1,k_2\}$.

Remark 1. *In the following discussion, we will make frequent use of a simple observation that can simply our notation and calculation. Since the choice of reference node is arbitrary, without loss of generality and after re-indexing the nodes, we can assume $\{0,k\}$ is the shared edge of G_1 and G_2 for some $k \neq 0$. This corresponds to a projection of the symmetric polytope onto one of the coordinate planes.*

Lemma 2. *For a connected graph G and one of its edges, $\{k_1,k_2\}$, every cell in the edge contraction subdivision $\mathcal{D}_{k_1,k_2}$ must contain both $\mathbf{e}_{k_1} - \mathbf{e}_{k_2}$ and $\mathbf{e}_{k_2} - \mathbf{e}_{k_1}$.*

Proof. As noted in Remark 1, without loss of generality, we can assume $\{0,k\}$ is the shared edge for some $k \neq 0$. In the follow, we consider the regular subdivision induced by $\omega = \omega_{0,k}$ and will show that $\pm\mathbf{e}_k \in C$ for all $C \in \mathcal{D}$.

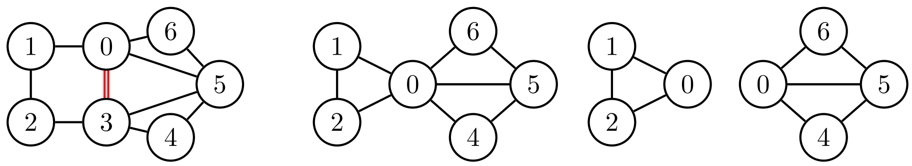
Fix a cell $C \in \mathcal{D}$, let $\hat{C}$ be the corresponding lower facet of $\hat{\nabla}_G$, let $\hat{\boldsymbol{\alpha}} = (\boldsymbol{\alpha}, 1) = (\alpha_1, \dots, \alpha_n, 1)$ be the upward pointing inner normal vector of $\hat{C}$, and let $h = \min\{\langle \hat{\boldsymbol{\alpha}}, \hat{\mathbf{x}} \rangle \mid \hat{\mathbf{x}} \in \hat{\nabla}_G\}$.

Suppose $\pm\mathbf{e}_k \notin C$, then there is a set of $n+1$ affinely independent points of the form $\mathbf{e}_i - \mathbf{e}_j$ with $\{i,j\} \neq \{0,k\}$ in C. By assumption, $\hat{\boldsymbol{\alpha}}$ is orthogonal to the affine span of this set. However, since $\omega(\mathbf{e}_i - \mathbf{e}_j) = 1$ for $\{i,j\} \neq \{0,k\}$, the affine span of $\hat{C}$ must be $\{(\mathbf{x}, 1) \mid \mathbf{x} \in \mathbb{R}^n\}$, and consequently its normal vector $\hat{\boldsymbol{\alpha}}$ must be $(0, \dots, 0, 1)$. Then for any $\mathbf{e}_i - \mathbf{e}_j \in C$,

$$\langle (\mathbf{e}_i - \mathbf{e}_j, \omega(\mathbf{e}_i - \mathbf{e}_j)), \hat{\boldsymbol{\alpha}} \rangle = 1 > 0 = \langle (\pm\mathbf{e}_k, \omega(\pm\mathbf{e}_k)), \hat{\boldsymbol{\alpha}} \rangle,$$

contradicting with the assumption that $\langle \hat{\boldsymbol{\alpha}}, \cdot \rangle$ minimizes on $\hat{C}$ over $\hat{\nabla}_G$. We can conclude then either $\mathbf{e}_k$ or $-\mathbf{e}_k$ must be in C.

Now suppose $\mathbf{e}_k \in C$ but $-\mathbf{e}_k \notin C$. then $\langle -\mathbf{e}_k, \boldsymbol{\alpha} \rangle > \langle \mathbf{e}_k, \boldsymbol{\alpha} \rangle$ which implies that $h = \langle \mathbf{e}_k, \boldsymbol{\alpha} \rangle < 0$. Since C is an n-dimensional cell, there is a set Δ of n affinely independent points in C of the form $\mathbf{e}_i - \mathbf{e}_j$ with $\{i,j\} \neq \{0,k\}$, i.e., $\{\mathbf{e}_i - \mathbf{e}_j - \mathbf{e}_k \mid \mathbf{e}_i - \mathbf{e}_j \in \Delta\}$ is a linearly independent set. Let A be the $n \times n$



(A) Graph G formed by two subgraphs sharing an edge

(B) Graph G' resulted from edge contraction

(C) Two subgraphs G'_1 and G'_2 of G'

FIGURE 2. Edge contraction on a graph

matrix whose rows are points in Δ as row vectors, and let $B = A - \mathbf{1}\mathbf{e}_k^\top$, which is nonsignular. Recall that C is the projection of a lower facet of $\hat{\nabla}_G$ defined by the inner normal vector $\hat{\boldsymbol{\alpha}} = (\boldsymbol{\alpha}, 1)$. Therefore,

$$\langle\, \mathbf{e}_i - \mathbf{e}_j \,,\, \boldsymbol{\alpha} \,\rangle + 1 = \langle\, \mathbf{e}_k \,,\, \boldsymbol{\alpha} \,\rangle \quad \text{for each } \mathbf{e}_i - \mathbf{e}_j \in \Delta,$$

which is equivalent to

$$B\boldsymbol{\alpha} = A\boldsymbol{\alpha} - \mathbf{1}\mathbf{e}_k^\top \boldsymbol{\alpha} = -\mathbf{1}.$$

We will show this contradicts with the assumption that $h < 0$.

Suppose A is singular, let $\mathbf{x}$ be a nonzero vector in its null space. Then $\mathbf{e}_k^\top \mathbf{x} \neq 0$, since $B\mathbf{x} = A\mathbf{x} - \mathbf{1}\mathbf{e}_k^\top \mathbf{x}$ cannot be zero. We can verify that $\boldsymbol{\alpha} = \mathbf{x}/\mathbf{e}_k^\top \mathbf{x}$, and thus

$$h = \langle\, \mathbf{e}_k \,,\, \boldsymbol{\alpha} \,\rangle = \mathbf{e}_k^\top \mathbf{x}/\mathbf{e}_k^\top \mathbf{x} = 1,$$

which contradicts with the assumption that $h < 0$.

On the other hand, if A is nonsingular, then without loss of generality, it is possible to re-index the nodes $\{0, 1, \ldots, n\} \setminus \{0, k\}$ so that for each $i \neq 0, k$, $\pm(\mathbf{e}_i - \mathbf{e}_j) \in \Delta$ implies $j > i$. With this arrangement, A is upper triangular and its diagonal entries are ± 1. Therefore, A is unimodular. Consequently, A^{-1} exists and is an integer matrix. Recall that $B\boldsymbol{\alpha} = A\boldsymbol{\alpha} - \mathbf{1}\mathbf{e}_k^\top \boldsymbol{\alpha} = -\mathbf{1}$, and $h = \mathbf{e}_k^\top \boldsymbol{\alpha}$. This equation can be written as

$$A\,\boldsymbol{\alpha} = (h-1)\,\mathbf{1}, \qquad \text{i.e.} \qquad \boldsymbol{\alpha} = (h-1)\,A^{-1}\,\mathbf{1},$$

which gives us the relation

$$h = \mathbf{e}_k^\top\, \boldsymbol{\alpha} = h(\mathbf{e}_k^\top\, A^{-1}\mathbf{1}) - \mathbf{e}_k^\top A^{-1}\mathbf{1} = y\,h - y$$

if we let $y = \mathbf{e}_k^\top A^{-1}\mathbf{1}$. The above equation implies that $y \neq 1$. Moreover, since A^{-1} is an integer matrix, $y \in \mathbb{Z}$. Therefore,

$$h = \frac{y}{y-1} \geq 0$$

contradicting with the assumption that $h = \mathbf{e}_k^\top \boldsymbol{\alpha} < 0$. That is, the assumption $\mathbf{e}_k \in C$ but $\mathbf{e}_k \notin C$ leads to a contradiction. We can therefore conclude that $\mathbf{e}_k \in C$ implies $-\mathbf{e}_k \in C$. By the same argument, it can be shown that $-\mathbf{e}_k \in C$ implies $\mathbf{e}_k \in C$. Hence, $\pm\mathbf{e}_k \in C$ and $\alpha_k = 0$. □

In the following, using the special edge contraction subdivision, we establish the link between ∇_G and $\nabla_{G /\!\!/ \{k_1, k_2\}}$.

3.1. **Two subgraphs sharing an edge.** We first consider the case where the target graph G consists of two sub-graphs sharing a single edge with the two corresponding vertices forming a cut set. Figure 2a shows an example of such a graph. That is, there are two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ such that $\mathcal{V}(G) = V_1 \cup V_2, \mathcal{E}(G) = E_1 \cup E_2$, and there is one edge $e = \{k_1, k_2\} \in \mathcal{E}(G)$ such that $V_1 \cap V_2 = \{k_1, k_2\}$ and $E_1 \cap E_2 = \{e\}$. The contraction $G' = G \mathbin{/\!\!/} e$, shown in Figure 2b, thus has a cut vertex, which allows us to consider the two separate graphs (Figure 2c).

Theorem 1. *For a connected graph G consisting of two subgraphs G_1 and G_2 sharing a single edge $e = \{k_1, k_2\}$, let $G_1' = G_1 \mathbin{/\!\!/} e$ and $G_2' = G_2 \mathbin{/\!\!/} e$. Then the cells in the edge contraction subdivision $\mathcal{D}_{k_1,k_2}$ of ∇_G induced by the contraction of G along $\{k_1, k_2\}$ are in one-to-one correspondence with pairs in $\mathcal{F}(\nabla_{G_1'}) \times \mathcal{F}(\nabla_{G_2'})$.*

Proof. As before, using the observation provided in Remark 1, we can assume $\{0, k\}$ is the shared edge for some $k \neq 0$. In addition, we assume the index k is chosen so that $i < k$ for all $i \in \mathcal{V}(G_1)$ and $k < j$ for all $j \in \mathcal{V}(G_2)$. That is, after renaming the nodes, we assume nodes $1, 2, \ldots, k-1$ are in G_1, nodes $k+1, \ldots, n$ are in G_2, and nodes $0, k$ are in both. We only need to consider the subdivision $\mathcal{D}_{0,k}$.

By definition, the adjacency polytopes $\nabla_{G_1'}$ and $\nabla_{G_2'}$ are full-dimensional polytopes in the subspaces $\mathbb{R}^{k-1} \times \{\mathbf{0}_{n-k+1}\}$ and $\{\mathbf{0}_k\} \times \mathbb{R}^{n-k}$ respectively. So their facets are of dimensions $k-2$ and $n-k-1$ respectively.

By Lemma 2, any cell C must contain both $\pm\mathbf{e}_k$, and therefore its upward pointing inner normal vector $\hat{\boldsymbol{\gamma}} = (\boldsymbol{\gamma}, 1) \in \mathbb{R}^{n+1}$ that defines the lower facet $\hat{C}$ of $\hat{\nabla}_G$ satisfies $\langle \pm\mathbf{e}_k \,,\, \boldsymbol{\gamma} \rangle = 0$. Thus $\hat{C}$ is contained in the hyperplane $\langle \cdot \,,\, \hat{\boldsymbol{\gamma}} \rangle = 0$, and

$$\boldsymbol{\gamma} = (\boldsymbol{\alpha}, 0, \boldsymbol{\beta}) \quad \text{for some } \boldsymbol{\alpha} \in \mathbb{R}^{k-1} \text{ and } \boldsymbol{\beta} \in \mathbb{R}^{n-k}.$$

Let C_1 and C_2 be the sets of projections of points in C in $\mathbb{R}^{k-1} \times \{\mathbf{0}_{n-k+1}\}$ and $\{\mathbf{0}_k\} \times \mathbb{R}^{n-k}$ respectively. We will show the nonzero points in C_1 and C_2 define facets of $\nabla_{G_1'}$ and $\nabla_{G_2'}$ respectively. Since $\gamma = (\boldsymbol{\alpha}, 0, \boldsymbol{\beta})$, the equation (2) which defines the lower facet $\hat{C}$ implies

$$\langle \mathbf{a} \,,\, (\boldsymbol{\alpha}, \mathbf{0}_{n-k+1}) \rangle = -1 \qquad \text{for all } \mathbf{0} \neq \mathbf{a} \in C_1, \tag{6}$$
$$\langle \mathbf{b} \,,\, (\mathbf{0}_k, \boldsymbol{\beta}) \rangle = -1 \qquad \text{for all } \mathbf{0} \neq \mathbf{b} \in C_2, \tag{7}$$
$$\langle \mathbf{e}_i - \mathbf{e}_j \,,\, (\boldsymbol{\alpha}, \mathbf{0}_{n-k+1}) \rangle \geq -1 \qquad \text{for all } \{i, j\} \in \mathcal{E}(G_1'),$$
$$\langle \mathbf{e}_i - \mathbf{e}_j \,,\, (\mathbf{0}_k, \boldsymbol{\beta}) \rangle \geq -1 \qquad \text{for all } \{i, j\} \in \mathcal{E}(G_2'),$$

and there are at least $n-1$ equalities in total in this system (since the two equalities associated with $\pm\mathbf{e}_k$ are removed). Moreover, since $\dim(\operatorname{conv}(C)) = n$, the corresponding points can be chosen to be linearly independent as vectors. Therefore, there are at least $k-1$ and $n-k$ equalities among (6) and (7) respectively. By Lemma 1, the vectors $(\boldsymbol{\alpha}, \mathbf{0}_{n-k+1})$ and $(\mathbf{0}_k, \boldsymbol{\beta})$ must define facets in $\nabla_{G_1'}$ and $\nabla_{G_2'}$ respectively. That is, if we identify each cell with its corresponding upward-pointing inner normal vector, then the map

$$\gamma \;\mapsto\; (\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \mathbb{R}^{k-1} \times \mathbb{R}^{n-k}$$

sends each cell in $\mathcal{D}_{0,k}$ to a pair of facets of $\nabla_{G_1'}$ and $\nabla_{G_2'}$ respectively.

We now simply have to show that this map has an inverse. Suppose F_1' and F_2' are two facets of $\nabla_{G_1'}$ and $\nabla_{G_2'}$ respectively. We will construct a corresponding cell in $\mathcal{D}_{0,k}$. Let $\mathbf{a}_1, \ldots, \mathbf{a}_{m_1'} \in \phi(\mathcal{E}(G_1'))$ and $\mathbf{b}_1, \ldots, \mathbf{b}_{m_2'} \in \phi(\mathcal{E}(G_2'))$ be the points defining F_1' and F_2' respectively for some $m_1' \geq k-1$ and $m_2' \geq n-k$. Then by Lemma 1, there are vectors $\boldsymbol{\alpha} \in \mathbb{R}^{k-1}$ and $\boldsymbol{\beta} \in \mathbb{R}^{n-k}$ such that

$$\begin{aligned} \langle \mathbf{a}_i \,,\, (\boldsymbol{\alpha}, \mathbf{0}_{n-k+1}) \rangle &= -1 && \text{for all } i = 1, \dots, m_1', \\ \langle \mathbf{e}_i - \mathbf{e}_j \,,\, (\boldsymbol{\alpha}, \mathbf{0}_{n-k+1}) \rangle &\geq -1 && \text{for all } \{i,j\} \in \mathcal{E}(G_1'), \\ \langle \mathbf{b}_i \,,\, (\mathbf{0}_k, \boldsymbol{\beta}) \rangle &= -1 && \text{for all } i = 1, \dots, m_2', \\ \langle \mathbf{e}_i - \mathbf{e}_j \,,\, (\mathbf{0}_k, \boldsymbol{\beta}) \rangle &\geq -1 && \text{for all } \{i,j\} \in \mathcal{E}(G_2'). \end{aligned}$$

Define

$$\boldsymbol{\gamma} = (\boldsymbol{\alpha}, 0, \boldsymbol{\beta}) \in \mathbb{R}^{k-1+1+n-k} = \mathbb{R}^n.$$

We will verify that $\hat{\boldsymbol{\gamma}} = (\boldsymbol{\gamma}, 1) \in \mathbb{R}^{n+1}$ defines a lower facet of $\hat{\nabla}_G$. Let

$$\begin{aligned} C_1 &= \{(\mathbf{a}_i, p, \mathbf{0}) \mid p \in \{-1,0,1\} \,,\, i = 1, \dots, m_1'\} \cap \phi_0(\mathcal{E}(G)), \\ C_2 &= \{(\mathbf{0}, p, \mathbf{b}_i) \mid p \in \{-1,0,1\} \,,\, i = 1, \dots, m_2'\} \cap \phi_0(\mathcal{E}(G)). \end{aligned}$$

Then

$$\begin{aligned} \langle (\mathbf{a}_i, p, \mathbf{0}) \,,\, (\boldsymbol{\alpha}, 0, \boldsymbol{\beta}) \rangle + 1 &= \langle \mathbf{a}_i \,,\, \boldsymbol{\alpha} \rangle + 1 = -1 + 1 = 0 && \text{for each } (\mathbf{a}_i, p, \mathbf{0}) \in C_1, \\ \langle (\mathbf{0}, p, \mathbf{b}_i) \,,\, (\boldsymbol{\alpha}, 0, \boldsymbol{\beta}) \rangle + 1 &= \langle \mathbf{b}_i \,,\, \boldsymbol{\beta} \rangle + 1 = -1 + 1 = 0 && \text{for each } (\mathbf{0}, p, \mathbf{b}_i) \in C_2. \end{aligned}$$

Moreover, $\langle (\pm \mathbf{e}_k, 0) \,,\, \hat{\boldsymbol{\gamma}} \rangle = 0$. Let

$$C = C_1 \cup C_2 \cup \{\pm \mathbf{e}_k\},$$

then $\hat{C}$ is contained in the hyperplane defined by $\langle \cdot \,,\, \hat{\gamma} \rangle = 0$. For points in $\phi(\mathcal{E}(G)) \setminus \{\pm \mathbf{e}_k\}$, direct computation confirms that

$$\begin{aligned} \langle \pm(\mathbf{e}_i - \mathbf{e}_j) \,,\, \boldsymbol{\gamma} \rangle + 1 &= \langle \pm(\mathbf{e}_i - \mathbf{e}_j) \,,\, (\boldsymbol{\alpha}, \mathbf{0}_{n-k+1}) \rangle + 1 \geq 0 && \text{for any } i,j < k, \\ \langle \pm(\mathbf{e}_i - \mathbf{e}_j) \,,\, \boldsymbol{\gamma} \rangle + 1 &= \langle \pm(\mathbf{e}_i - \mathbf{e}_j) \,,\, (\mathbf{0}_k, \boldsymbol{\beta}) \rangle + 1 \geq 0 && \text{for any } i,j > k, \\ \langle \pm(\mathbf{e}_i - \mathbf{e}_k) \,,\, \boldsymbol{\gamma} \rangle + 1 &= \langle \pm \mathbf{e}_i \,,\, (\boldsymbol{\alpha}, \mathbf{0}_{n-k+1}) \rangle + 1 \geq 0 && \text{for any } 0 < i < k, \\ \langle \pm(\mathbf{e}_j - \mathbf{e}_k) \,,\, \boldsymbol{\gamma} \rangle + 1 &= \langle \pm \mathbf{e}_j \,,\, (\mathbf{0}_k, \boldsymbol{\beta}) \rangle + 1 \geq 0 && \text{for any } j > k, \\ \langle \pm(\mathbf{e}_j - \mathbf{e}_0) \,,\, \boldsymbol{\gamma} \rangle + 1 &= \langle \pm \mathbf{e}_j \,,\, (\mathbf{0}_k, \boldsymbol{\beta}) \rangle + 1 \geq 0 && \text{for any } j > k. \end{aligned}$$

Therefore, $\operatorname{conv}(C)$ is a projection of a lower face. By construction,

$$|C| = |C_1| + |C_2| + |\{\pm \mathbf{e}_k\}| \geq (k-1) + (n-k) + 2 = n+1,$$

and these points have affinely indepedent projections in $\mathbb{R}^{k-1} \times \{\mathbf{0}\}$, $\{\mathbf{0}_{k-1}\} \times \mathbb{R} \times \{\mathbf{0}_{n-k}\}$, or $\{\mathbf{0}_k\} \times \mathbb{R}^{n-k}$. So,

$$\dim(\operatorname{conv}(C)) \geq k - 1 + 1 + n - k = n.$$

Therefore, C must be a projection of a lower facet of $\hat{\nabla}_G$ and hence a cell in $\mathcal{D}_{0,k}$. □

The theorem above establishes a bijection between cells in $\mathcal{D}_{k_1,k_2}$ and pairs of faces in $\mathcal{F}(\nabla_{G_1'})$ and $\mathcal{F}(\nabla_{G_2'})$. For later reference, this bijection will be denoted by $q_{k_1,k_2} : \mathcal{D}_{k_1,k_2} \to \mathcal{F}(\nabla_{G_1'}) \times \mathcal{F}(\nabla_{G_2'})$ and given by

$$q_{k_1,k_2}(C) = (F_1, F_2),$$

where F_1 and F_2 are simply the convex hull of the projections of C in the coordinate-subspaces in which $\nabla_{G_1'}$ and $\nabla_{G_2'}$ are full-dimensional.

Remark 2. *Note that q_{k_1,k_2} is only a bijection between the set of cells in $\mathcal{D}_{k_1,k_2}$ and the set $\mathcal{F}(\nabla_{G_1'}) \times \mathcal{F}(\nabla_{G_2'})$. The points in C themselves may not be in one-to-one correspondence with vertices in F_1 and F_2. In general, the projection that maps points in C to vertices of F_1 and F_2 may be not be one-to-one. This is a reflection of the fact that the edge-contraction operation may map multiple edges to the same edge, since we only allow simple graphs (graphs with no multiple edges and loops).*

In certain applications (e.g., the root counting problem for algebraic Kuramoto equations), the normalized volume of ∇_G is of great importance. With the subdivision $\mathcal{D}_{k_1,k_2}$ of ∇_G, the normalized volume of ∇_G can be computed as the sum of the normalized volume of each cell. The normalized volume of a minimum cell (a simplicial cell) can be computed directly. Indeed, the normalized volume in this case will always be 2.

Theorem 2. *Suppose G is a graph consisting of two subgraphs G_1 and G_2 sharing a single edge $e = \{k_1, k_2\}$, with $G_1' = G_1 /\!\!/ e$ and $G_2' = G_2 /\!\!/ e$. Let C be a cell in the edge contraction subdivision $\mathcal{D}_{k_1,k_2}$ with $q_{k_1,k_2}(C) = (F_1, F_2)$ for some facets F_1 and F_2 of $\nabla_{G_1'}$ and $\nabla_{G_2'}$ respectively. If C is simplicial, then F_1 and F_2 are both simplicial, and*

$$\operatorname{nvol}(C) = 2.$$

Proof. Without loss of generality, we still adopt the convention that $\{0, k\}$ is the shared edge, and $i \le k$ for all $i \in \mathcal{V}(G_1)$ and $k \le j$ for all $j \in \mathcal{V}(G_2) \setminus \{0\}$. With this convention, $\nabla_{G_1'}$ and $\nabla_{G_2'}$ are embedded in $\mathbb{R}^{k-1} \times \{\mathbf{0}_{n-k+1}\}$ and $\{\mathbf{0}_k\} \times \mathbb{R}^{n-k}$ respectively.

Let $C \in \mathcal{D}_{0,k}$ be a simplicial cell, and let $(F_1, F_2) = q_{0,k}(C)$, then F_1 and F_2, being facets of $\nabla_{G_1'}$ and $\nabla_{G_2'}$, are of dimensions $k-2$ and $n-k-1$ respectively. Suppose either F_1 or F_2 is not simplicial, then the combined total number of vertices is at least

$$(k - 2 + 1) + (n - k - 1 + 1) + 1 = n.$$

Since these points are nonzero projections of points in C, so C contains convex independent set of n points $\mathbf{x}_1, \dots, \mathbf{x}_n \in \phi(\mathcal{E}(G))$ with nonzero projections in $\mathbb{R}^{k-1} \times \{\mathbf{0}_{n-k+1}\}$ or $\{\mathbf{0}_k\} \times \mathbb{R}^{n-k}\}$. In addition, C contains two points $\pm\mathbf{e}_k$, which are in the fiber over $\mathbf{0}$ with respect to either projection, and thus $\pm\mathbf{e}_k \notin \operatorname{conv}\{\mathbf{x}_1, \dots, \mathbf{x}_n\}$. Therefore, C contain at least $n+2$ points, and none of them is a interior point. This contradicts with the assumption that C is simplicial. Therefore, we can conclude that if C is simplicial, then F_1 and F_2 must also be simplicial.

Under this assumption, there are $\mathbf{a}_1, \dots, \mathbf{a}_{k-1} \in \phi(\mathcal{E}(G_1')) \subset \mathbb{R}^{k-1} \times \{\mathbf{0}_{n-k+1}\}$ and $\mathbf{b}_1, \dots, \mathbf{b}_{n-k} \in \phi(\mathcal{E}(G_2')) \subset \{\mathbf{0}_k\} \times \mathbb{R}^{n-k}$ so that

$$F_1 = \operatorname{conv}\{\mathbf{a}_1, \dots, \mathbf{a}_{k-1}\} \qquad \text{and} \qquad F_2 = \operatorname{conv}\{\mathbf{b}_1, \dots, \mathbf{b}_{n-k}\}.$$

Moreover, with this embedding, the projection of vertices in C in $\mathbb{R}^{k-1} \times \{\mathbf{0}_{n-k+1}\}$ are exactly the points $\mathbf{a}_1, \dots, \mathbf{a}_{k-1}$. Similarly, the projection of vertices in C in

$\{\mathbf{0}_k\} \times \mathbb{R}^{n-k}$ are $\mathbf{b}_1, \dots, \mathbf{b}_{k-1}$. Therefore,

$$\operatorname{nvol}(C) = \det \begin{bmatrix} \mathbf{a}_1^\top & & \\ \vdots & & \\ \mathbf{a}_{k-1}^\top & & \\ & 2\mathbf{e}_k^\top & \\ & & \mathbf{b}_1^\top \\ & & \vdots \\ & & \mathbf{b}_{n-k}^\top \end{bmatrix} = 2 \det \begin{bmatrix} \mathbf{a}_1^\top \\ \vdots \\ \mathbf{a}_{k-1}^\top \end{bmatrix} \det \begin{bmatrix} \mathbf{b}_1^\top \\ \vdots \\ \mathbf{b}_{n-k}^\top \end{bmatrix},$$

which is precisely $2 \cdot \operatorname{nvol}_{k-1}(F_1') \cdot \operatorname{nvol}_{n-k}(F_2')$.

As shown in [4], a simplicial facet of an adjacency polytope has normalized volume 1. So the above equality simplifies to $\operatorname{nvol}(C) = 2$. □

3.2. **Edge contraction in a single graph.** We now apply the results from Section 3.1 to the edge contraction in a single graph G. Namely, we consider G as a union of two graphs, $G_1 = G$ and G_2 which has exactly one edge $e = \{k_1, k_2\}$. Then Theorems 1 and 2 imply the following corollaries.

Corollary 1. *Let $e = \{k_1, k_2\}$ be an edge of a connected graph G. Let $G' = G /\!\!/ e$. Then the cells in the edge contraction subdivision $\mathcal{D}_{k_1,k_2}$ of ∇_G induced by the contraction of G along the edge e are in one-to-one correspondence with the facets of $\nabla_{G'}$.*

Corollary 2. *Let $e = \{k_1, k_2\}$ be an edge of a connected graph G. Let $G' = G /\!\!/ e$. Suppose C is a cell in the edge contraction subdivision $\mathcal{D}_{k_1,k_2}$ with $q_{k_1,k_2}(C) = (F, \varnothing)$ for some facet F of $\nabla_{G'}$. If C is simplicial, then F is simplicial, and*

$$\operatorname{nvol}(C) = 2.$$

4. Conclusion

Adjacency polytopes, a.k.a. symmetric edge polytopes, are convex polytopes associated with connected simple graphs that have found important applications in several seemingly independent fields. The set of facets and regular subdivisions of an adjacency polytope are particularly important in certain applications (e.g. the study of algebraic Kuramoto equations). Recent works established explicit descriptions of the facets and subdivisions of many families of graphs including trees, cycles, wheels, and bipartite graphs [3, 4, 5]. The general description for facets and subdivision for arbitrary connected graphs remains an important open problem.

In this paper, we took one step toward a recursive approach for understanding the geometric structure of adjacency polytopes associated with large and complex graphs by considering the effect of an edge-contraction of a graph on the subdivisions of the corresponding adjacency polytope. In particular, we showed that an edge-contraction on a graph G naturally induces a special regular subdivision of ∇_G whose cells are in one-to-one correspondence with facets or product of facets of the adjacency polytope(s) associated with the smaller resulting graph(s). Combined with the existing understanding of the facet structures for adjacency polytopes associated with trees, cycles, wheels, bipartite graphs, etc., this correspondence can shed light on the regular subdivisions of more complicated graphs formed by gluing these graphs along edges.

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