COMPUTING VOLUMES OF ADJACENCY POLYTOPES VIA DRACONIAN SEQUENCES

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Abstract. Adjacency polytopes of graphs appear naturally in the study of nonlinear emergent phenomenon in complex networks. The "type-PV" adjacency polytope, also known as a symmetric edge polytope, arises in the study of Kuramoto equations. The "type-PQ" adjacency polytope of the graph G, which we denote by ∇_G^{PQ} which is the focus of this work, encodes rich combinatorial information about power-flow solutions in sparse power networks that are studied in electric engineering. Of particular importance is the normalized volume of such an adjacency polytope, which provides an upper bound on the number of distinct power-flow solutions.

In this article we show that the problem of normalized volumes for the type-PQ adjacency polytopes can be rephrased as counting $D(G)$-draconian sequences where $D(G)$ is a certain bipartite graph associated to the network. We provide recurrences for all networks with connectivity as most 1 and, for 2-connected graphs, we give, under an additional mild restriction, recurrences for subdividing an edge and taking the join of an edge with a new vertex. Together, these recurrences imply a simple, non-recursive formula for the normalized volume of ∇_G^{PQ} when G is part of a large class of outerplanar graphs; we conjecture that the formula holds for all outerplanar graphs. Explicit formulas for several other (non-outerplanar) classes are given. Further, we identify several important classes of graphs G which are planar but not outerplanar worth additional study.

Contents

1. Introduction and Background

Let $G = (V(G), E(G))$ be a simple graph on $[N] = \{1, \dots, N\}$. We use $e_1, \dots, e_N$ to denote the standard basis vectors of $\mathbb{R}^N$. The *PQ-type adjacency polytope* of G is defined to be

$$\nabla_G^{\mathrm{PQ}} = \mathrm{conv}\{(e_i, e_j) \in \mathbb{R}^{2N} \mid ij \in E(G) \text{ or } i = j\}$$

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where $\operatorname{conv}(S)$ denotes the convex hull of elements of S. Its *normalized volume*, $\operatorname{NVol}(\nabla_G^{\mathrm{PQ}}) = \dim(\nabla_G^{\mathrm{PQ}})!\operatorname{vol}(\nabla_G^{\mathrm{PQ}})$ where $\operatorname{vol}(P)$ is the relative volume of P, is always a positive integer.

The study of PQ-type adjacency polytopes were introduced in [2], motivated by the engineering problem known as *power-flow study* (or *load-flow study*). This study models the balance of electric power on a network of power generation or delivery "buses". Of particular importance are the alternating current (AC) variations, which produce nonlinear equations that are notoriously difficult to analyze. In the AC model for a power network with buses labeled as $1, \ldots, N$, the voltage on each bus is expressed as a complex variable $v_i = x_i + \mathbf{i} y_i$ whose absolute value represents the voltage magnitude and whose argument encodes the phase of the AC experienced on the bus. The interaction among buses is modeled by a graph G whose nodes represent the buses and whose edges represent the junctions. Kirchhoff's circuit laws give rise to an idealized balancing condition for the power injected, power generated, and power consumed on each bus, which can be expressed as the system of nonlinear equations

$$S_i = \sum_{j=1}^{N} \overline{Y}_{ij} v_i \overline{v}_j \quad \text{for } i = 2, \ldots, N, \tag{1}$$

where $S_i = P_i + \mathbf{i} Q_i$ is a complex representation of the real and reactive power, Y_{ij}, known as *nodal admittance*, describes the connection between the i and j buses, and $\overline{Y}_{ij}$ and $\overline{v}_j$ denote the complex conjugate of Y_{ij} and v_j respectively. By dropping the conjugate constraints between v_i and $\overline{v}_i$, we obtained the algebraic version of this system, known as the algebraic power-flow equations. It was shown that the maximum number of nontrivial complex solutions this system has is bounded by the normalized volume of ∇_G^{PQ}.

We take care to call the adjacency polytopes within this paper PQ-type, since a related construction is sometimes called an adjacency polytope; see, for example, [4, 5]. This alternate construction, motivated by counting equilibrium solutions to a network of interconnected oscillators, relies on a particular change of variables that is not available here. In engineering terms, this alternate construction arises from PV-type buses.

In this article we show that the normalized volume of ∇_G^{PQ} can be described in terms of sequences of nonnegative integers related to the *Dragon Marriage Problem*: a variant of Hall's Matching Theorem that has far-reaching applications and spawned the study of generalized permutohedra [9, 10]. We establish this relationship in Section 2 and show how it can be immediately exploited to compute normalized volumes of some PQ-type adjacency polytopes when G is nontrivial.

We explore this connection more deeply in Section 3 where we establish several recurrences. Namely, we provide recurrences for all graphs with connectivity at most 1, that is, any graph that is disconnected or has a cut-vertex. These directly imply a simple formula for $\operatorname{NVol}(\nabla_G^{\mathrm{PQ}})$ whenever G is a forest.

Sections 3.1 and 3.2 consider two operations on a 2-connected graph: subdivision of an edge e and replacing e with the join of e and a new vertex. Under mild conditions, these operations lead to the following two recurrences that are stated simply but nontrivial to prove.

Theorem 3.10 (Subdivision recurrence)**.** Let G be a connected graph on $[N]$ with an edge $e = uv$. Denote by $G : e$ the graph obtained by subdividing e. If $\deg_G(u) = 2$ then

$$\operatorname{NVol}(\nabla_{G:e}^{\mathrm{PQ}}) = 2\operatorname{NVol}(\nabla_G^{\mathrm{PQ}}) + \operatorname{NVol}(\nabla_{G\setminus e}^{\mathrm{PQ}}).$$

Theorem 3.18 (Triangle Recurrence)**.** Let G be a connected graph on $[N]$ with an edge $e = uv$. Denote by $G\triangle e$ the graph on $[N+1]$ with edge set $E(G) \cup \{\{u, N+1\}, \{v, N+1\}\}$. If $\deg_G(u) = 2$, then

$$\mathrm{NVol}(\nabla^{\mathrm{PQ}}_{G\triangle e}) = 3\,\mathrm{NVol}(\nabla^{\mathrm{PQ}}_{G}).$$

Section 3 concludes by applying the recurrences to establish a closed, non-recursive formula for $\mathrm{NVol}(\nabla^{\mathrm{PQ}}_{G})$ for a large class of outerplanar graphs; we conjecture that this formula holds for all outerplanar graphs. The final section addresses several classes of graphs which are planar but not outerplanar. First, we give results for a complete bipartite graph where one partite set has just two elements. Then we consider the classes of *wheel graphs* and *series-parallel graphs*, which are natural points of further study and will likely require a refinement of the techniques within this article or alternate techniques altogether.

2. Notation, background, and translating to draconian sequences

Before we prove our results, we will establish assorted notation that will be needed throughout this work. Additional notation will be introduced as needed. First, if e is an edge of G with endpoints u and v, we will write $e = uv$ or $e = vu$ whenever possible. When additional clarity is helpful we may alternately write $e = \{u, v\}$ or $e = \{v, u\}$.

If $X \subseteq V(G)$ then we use $G - X$ to denote the graph obtained from deleting the vertices of X as well as any edge that is incident to some vertex in X. If $X = \{v\}$ then we will just write $G - v$. Similarly, if S is a set of edges then we use $G \setminus S$ to denote the graph with the edges in S deleted; if $S = \{e\}$ then we just write $G - e$. If $X \subseteq V(G)$, then we use $G[X]$ to denote the subgraph of G induced by X. Lastly, if H is a graph then we use $G \vee H$ to denote the *join* of G and H, that is, the graph with vertex set $V(G) \cup V(H)$ and edge set

$$E(G) \cup E(H) \cup \{uv \mid u \in V(G), v \in V(H)\}.$$

For a positive integers M, N, let K_N denote the complete graph on $[N]$ and let $K_{M,\overline{N}}$ denote the complete bipartite graph with partite sets $[M]$ and $[\overline{N}] = \{\overline{1}, \dots, \overline{N}\}$. Let $\mathcal{N}_G(v)$ denote the set of vertices of G adjacent to v. Keeping this notation in mind, we may now begin in earnest.

In [10], Postnikov investigated the *Dragon Marriage Problem*, providing a generalization of Hall's Matching Theorem for bipartite graphs. In the Dragon Marriage Problem, a small medieval village is home to n grooms and $n+1$ brides, some pairs of whom would form compatible marriages. Suppose we know all pairs of compatible grooms and brides. One day, a dragon arrives in the village and kidnaps a bride. What compatibility conditions among the original set of grooms and brides will guarantee that those who remain can still be entirely paired by compatible marriages? In graph-theoretic terms, and more generally, consider an X, Y-bigraph G such that $|Y| = |X| + 1$. What are necessary and sufficient conditions on G so that $G - y$ has a perfect matching regardless of choice of $y \in Y$? The answer relies on the following.

Definition 2.1. Let $G \subseteq K_{N,\overline{N}}$. Call $(a_1, \dots, a_N) \in \mathbb{Z}^N_{\geq 0}$ a *G-draconian sequence* if $\sum a_i = N - 1$ and, for any $1 \leq i_1 < i_2 < \cdots < i_k \leq N$,

$$a_{i_1} + \cdots + a_{i_k} < \left| \bigcup_{j=1}^{k} \mathcal{N}_G(i_j) \right|. \tag{2}$$

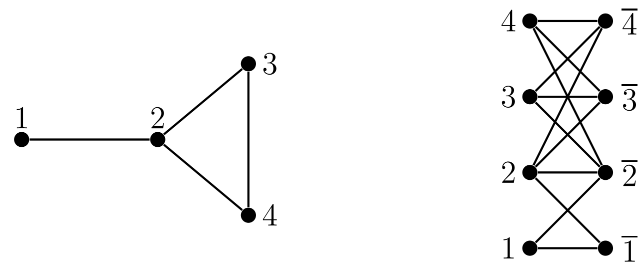

FIGURE 1. A graph G, left, and its corresponding bipartite graph $D(G)$, right.

We will say that a sequence satisfying (2) satisfies the *G-draconian inequality corresponding to* $i_1, \dots, i_k$.

Postnikov proved [10, Proposition 5.4 and Definition 9.2] that a matching that covers X exists exactly when a G-draconian sequence exists. He then goes on to compute volumes of certain polyhedra as sums over the set of G-draconian sequences. At the moment, it may be completely unclear how draconian sequences are useful to us; the rest of this section is dedicated to clarifying the connection.

Definition 2.2. Given a graph $G \subseteq K_{M,\overline{N}}$, let Q_G denote the *root polytope*

$$Q_G = \operatorname{conv}\{e_i - e_{\overline{j}} \mid (i, \overline{j}) \in E(G)\} \subseteq \mathbb{R}^M \times \mathbb{R}^{\overline{N}},$$

where $\mathbb{R}^{\overline{N}}$ denotes the real vector space with standard basis vectors $e_{\overline{1}}, \dots, e_{\overline{N}}$.

It turns out that we can describe ∇_G^{PQ} as a root polytope for an appropriate choice of graph.

Definition 2.3. Let G be a simple graph on $[N]$. Define $D(G)$ to be the subgraph of $K_{N,\overline{N}}$ with edges $\{i, \overline{i}\}$ for each $i \in [N]$ and $\{i, \overline{j}\}$ and $\{j, \overline{i}\}$ for each edge ij in G.

As an example, let G be the graph on [4] with edges 12, 23, 34, 24. Then $D(G)$ is the bipartite graph with vertices $\{1, 2, 3, 4, \overline{1}, \overline{2}, \overline{3}, \overline{4}\}$ and edges $1\overline{1}$, $1\overline{2}$, $2\overline{1}$, $2\overline{2}$, $2\overline{3}$, $2\overline{4}$, $3\overline{2}$, $3\overline{3}$, $3\overline{4}$, $4\overline{2}$, $4\overline{3}$, and $4\overline{4}$. See Figure 1 for an illustration.

Identifying $e_{\overline{i}}$ in $\mathbb{R}^{\overline{N}}$ with $-e_{N+i}$ in $\mathbb{R}^{2N}$ is a unimodular equivalence; thus, we have the following simple but important result.

Lemma 2.4. For all G, ∇_G^{PQ} is unimodularly equivalent to $Q_{D(G)}$. □

We now list two more theorems from [10]. In the first, $\sum$ denotes the Minkowski sum of polytopes and, given $S \subseteq [N]$, $\Delta_S = \operatorname{conv}\{e_i \mid i \in S\}$. It is also written to reflect our particular context and does not quite capture the full strength of the original statement. These two theorems are the last pieces needed to prove the main result of this section: Theorem 2.8.

Theorem 2.5 ([10, Theorem 12.2])**.** Let G be a graph on $[N]$ for which $D(G)$ is connected and let

$$P^-_{D(G)} = \left\{ x \in \mathbb{R}^N \mid x + \Delta_{[N]} \subseteq \sum_{i=1}^{N} \Delta_{\mathcal{N}_{D(G)}(i)} \right\}.$$

Then

$$\operatorname{NVol}(Q_{D(G)}) = |P^-_{D(G)} \cap \mathbb{Z}^N|.$$

As written, Theorem 2.5 relies on $D(G)$ being connected. Fortunately, the connectedness of G is equivalent to the connectedness of $D(G)$. We will use this fact occasionally so we present it as a lemma, although its proof is straightforward enough that we omit it.

Lemma 2.6. For any simple graph G, G is connected if and only if $D(G)$ is connected. □

Since we are primarily working with $D(G)$ rather than G directly, we let $\mathfrak{D}(G)$ denote the set of $D(G)$-draconian sequences.

Theorem 2.7 ([10, Theorem 11.3])**.** Let G be any graph. Then $|P^{-}_{D(G)} \cap \mathbb{Z}^N| = |\mathfrak{D}(G)|$.

Theorem 2.8. For any connected graph G on $[N]$, $\mathrm{NVol}(\nabla^{\mathrm{PQ}}_G) = |\mathfrak{D}(G)|$.

Proof. Lemma 2.6 assures us that $D(G)$ is connected. By Lemma 2.4, $\mathrm{NVol}(\nabla^{\mathrm{PQ}}_G) = \mathrm{NVol}(Q_{D(G)})$. Applying Theorem 2.5 and Theorem 2.7 completes the proof. □

To illustrate, let G be the graph on $[4]$ with edges 12, 23, and 24. Here, we have $\mathcal{N}_{D(G)}(1) = \{\overline{1}, \overline{2}\}$, $\mathcal{N}_{D(G)}(2) = \{\overline{1}, \overline{2}, \overline{3}, \overline{4}\}$, $\mathcal{N}_{D(G)}(3) = \{\overline{2}, \overline{3}\}$ and $\mathcal{N}_{D(G)}(4) = \{\overline{2}, \overline{4}\}$. Theorem 2.8 tells us that $\mathrm{NVol}(\nabla^{\mathrm{PQ}}_G) = 8$ since

$$\begin{aligned}\mathfrak{D}(G) = &\{(0,3,0,0),(0,2,0,1),(1,1,1,0),(1,1,0,1),\\ &(1,0,1,1),(0,1,1,1),(0,2,1,0),(1,2,0,0)\}.\end{aligned}$$

It will be very helpful for us to explicitly state when a sequence is $D(G)$-draconian. The main difference is recognizing that for every vertex i of G, $\deg_{D(G)}(i) = 1 + \deg_G(i)$.

Definition 2.1 (Draconian sequences, rephrased)**.** Let G be a graph on $[N]$. Call $(a_1, \ldots, a_N) \in \mathbb{Z}^N_{\geq 0}$ a *$D(G)$-draconian sequence* if $\sum a_i = N - 1$ and, for any $1 \leq i_1 < \cdots < i_k \leq N$,

$$a_{i_1} + \cdots + a_{i_k} < \left| \bigcup_{j=1}^{k} \mathcal{N}_{D(G)}(i_j) \right| = \left| \{i_1, \ldots, i_k\} \cup \left(\bigcup_{j=1}^{k} \mathcal{N}_G(i_j) \right) \right|$$

This translates our computation of normalized volume to a purely combinatorial computation. The following simple observation will also be helpful at several points when proving the results in Section 3.

Remark 2.9. The normalized volume of ∇^{PQ}_G is invariant under permutation of vertices.

We now give a first nontrivial application of Theorem 2.8 to an infinite class of graphs.

Proposition 2.10. Let $N > 2$ and let $\mathcal{M}$ be any matching of size k in K_N. Then

$$\mathrm{NVol}(\nabla^{\mathrm{PQ}}_{K_N \setminus \mathcal{M}}) = \binom{2(N-1)}{N-1} - 2k.$$

Proof. Note that since $N > 2$, $K_N \setminus \mathcal{M}$ is connected. First consider $k = 0$. The $D(K_N)$-draconian sequences are the weak compositions of $N - 1$ into N parts, of which there are $\binom{2(N-1)}{N-1}$. When $k > 0$, the deletion of each edge uv in M prohibits two compositions: those whose entries are all 0 except for one, which is $N - 1$ and located at position u or v. □

Proposition 2.10 refers to a very specific class of graphs. The next section proves results that allow for much more flexibility.

3. Draconian Recurrences

One of the main purposes of this article is to establish several recurrences for $\mathrm{NVol}(\nabla_G^{\mathrm{PQ}})$, using what we collectively call *draconian recurrences*. Certain specific recurrences will be given their own names as we encounter them. For a simple first situation we consider the disjoint union of two graphs G and H, which we denote $G+H$. Since Theorem 2.8 only applies to connected graphs, we study their adjacency polytopes directly.

If $P \subseteq \mathbb{R}^n$ and $Q \subseteq \mathbb{R}^m$ are polytopes, each containing the origins 0_n, 0_m respectively, then their *free sum* is

$$P \oplus Q = \mathrm{conv}\{(P \times 0_m) \cup (0_n \times Q)\} \subseteq \mathbb{R}^{n+m}.$$

When P and Q are lattice polytopes, there is a convenient product formula we may invoke.

Theorem 3.1 ([3, Theorem 2])**.** Given full-dimensional convex polytopes $P \subseteq \mathbb{R}^n$ and $Q \subseteq \mathbb{R}^m$, if both P and Q contain the origin of their respective ambient spaces, then

$$\mathrm{NVol}(P \oplus Q) = \mathrm{NVol}(P)\,\mathrm{NVol}(Q).$$

While Theorem 3.1 insists that P and Q are full-dimensional, we may replace them with unimodularly equivalent polytopes $P' \subseteq \mathbb{R}^{n'} \cong \mathrm{aff}(P)$ and $Q' \subseteq \mathbb{R}^{m'} \cong \mathrm{aff}(P)$. Since unimodular equivalence preserves normalized volume, the conclusion of Theorem 3.1 remains true. This gives us the last piece we need to prove the following.

Proposition 3.2. If G and H are any two graphs, then

$$\mathrm{NVol}(\nabla_{G+H}^{\mathrm{PQ}}) = \mathrm{NVol}(\nabla_G^{\mathrm{PQ}})\,\mathrm{NVol}(\nabla_H^{\mathrm{PQ}}).$$

Proof. Let $|V(G)| = M$ and $|V(H)| = N$. If $(x_1, \dots, x_{2M}) \in \nabla_G^{\mathrm{PQ}}$ then, by construction,

$$\sum_{i=1}^{M} x_i = 1 \quad \text{and} \quad \sum_{i=M+1}^{2M} x_i = 1,$$

and similar is true for $(y_1, \dots, y_{2N}) \in \nabla_H^{\mathrm{PQ}}$. It follows that the polytopes

$$P = \{(x_2, \dots, x_M, x_{M+2}, \dots, x_{2M}) \mid (x_1, \dots, x_{2M}) \in \nabla_G^{\mathrm{PQ}}\}$$

and

$$Q = \{(x_2, \dots, x_N, x_{N+2}, \dots, x_{2N}) \mid (x_1, \dots, x_{2N}) \in \nabla_H^{\mathrm{PQ}}\}$$

are projections that are unimodularly equivalent to ∇_G^{PQ} and ∇_H^{PQ}, respectively. Thus, $\mathrm{NVol}(\nabla_G^{\mathrm{PQ}}) = \mathrm{NVol}(P)$ and $\mathrm{NVol}(\nabla_H^{\mathrm{PQ}}) = \mathrm{NVol}(Q)$. Here, P and Q contain the origins of their respective ambient spaces, so $\mathrm{NVol}(P \oplus Q) = \mathrm{NVol}(P)\,\mathrm{NVol}(Q) = \mathrm{NVol}(\nabla_G^{\mathrm{PQ}})\,\mathrm{NVol}(\nabla_H^{\mathrm{PQ}})$.

Label the vertices of $G+H$ using $[M+N]$ by adding M to every vertex label of H. Let $f : \mathbb{R}^{2M+2N} \to \mathbb{R}^{2M+2N}$ be the map sending $(x_1, \dots, x_{2M+2N})$ to $(x_{\sigma(1)}, \dots, x_{\sigma(2M+2N)})$ where

$$\sigma(i) = \begin{cases} i & \text{if } i \le M \text{ or } i \ge 2M+N+1 \\ i+M & \text{if } M+1 \le i \le M+N \\ i-N & \text{if } M+N+1 \le i \le 2M+N. \end{cases}$$

Since f only permutes coordinates it is a unimodular transformation. Moreover, the projection of $f(\nabla_{G+H}^{\mathrm{PQ}})$ obtained from dropping the first, $(M+1)$th, $(2M+1)$th, and $(2M+N)$th coordinates is a lattice-preserving transformation sending $\nabla_{G+H}^{\mathrm{PQ}}$ onto $P \oplus Q$. Therefore,

$$\mathrm{NVol}(\nabla_{G+H}^{\mathrm{PQ}}) = \mathrm{NVol}(f(\nabla_{G+H}^{\mathrm{PQ}})) = \mathrm{NVol}(P \oplus Q) = \mathrm{NVol}(P)\,\mathrm{NVol}(Q) = \mathrm{NVol}(\nabla_G^{\mathrm{PQ}})\,\mathrm{NVol}(\nabla_H^{\mathrm{PQ}}),$$

proving the result. $\square$

In light of Proposition 3.2, we will focus for the rest of this section on graphs that are connected unless explicitly stated otherwise. Restricting to when G is connected allows us to use Theorem 2.8 and therefore we study the sets $\mathfrak{D}(G)$ directly rather than relying on properties of their polytopes.

Recall that a graph G is *k-connected* if for any set X of vertices, $|X| < k$, the subgraph $G - X$ is connected. A *block* of a graph G is an inclusion-maximal 2-connected subgraph of G.

Theorem 3.3. Suppose G is a connected graph with a block B containing the cut-vertex v. Setting $B' = G[(V(G) \setminus V(B)) \cup \{v\}]$ we have

$$\mathrm{NVol}(\nabla_G^{\mathrm{PQ}}) = \mathrm{NVol}(\nabla_B^{\mathrm{PQ}})\,\mathrm{NVol}(\nabla_{B'}^{\mathrm{PQ}}).$$

Proof. By Remark 2.9 we may assume without loss of generality that the cut-vertex is 1, that $V(B) = [M]$, and that $V(B') = \{1, M+1, \ldots, N\}$. We claim that the map

$$f : \mathfrak{D}(B) \times \mathfrak{D}(B') \to \mathfrak{D}(G)$$

which sends $\big((c_1, c_2, \ldots, c_M), (c'_1, c'_{M+1}, \ldots, c'_N)\big)$ to

$$(d_1, \ldots, d_N) = (c_1 + c'_1, c_2, \ldots, c_M, c'_{M+1}, \ldots, c'_N)$$

is a well-defined bijection.

For notational convenience set $c = (c_1, c_2, \ldots, c_M)$ and $c' = (c'_1, c'_{M+1}, \ldots, c'_N)$. Since $c \in \mathfrak{D}(B)$ and $c' \in \mathfrak{D}(B')$, we know $\sum c_i = M - 1$ and $\sum c'_i = N - M$. Thus, the sum of entries in $f(c, c')$ is $N - 1$, one of the requirements for being $D(G)$-draconian. Now pick any sequence $1 \leq i_1 < \cdots < i_k \leq N$. If $i_k < M$ or $M < i_1$ then the corresponding $D(G)$-draconian inequality automatically holds. So, suppose there is some positive $1 \leq \ell < k$ for which

$$i_1 < \cdots < i_\ell \leq M < i_{\ell+1} < \cdots < i_k.$$

If $1 < i_1$ then

$$\begin{aligned} d_{i_1} + \cdots + d_{i_j} &= c_{i_1} + \cdots + c_{i_\ell} + c'_{i_\ell+1} + \cdots + c'_k \\ &< \left| \bigcup_{j=1}^{\ell} \mathcal{N}_{D(B)}(i_j) \right| + \left| \bigcup_{j=\ell+1}^{k} \mathcal{N}_{D(B')}(i_j) \right| - 1. \end{aligned}$$

Since B and B' share just a single vertex, we have that

$$\left| \bigcup_{j=1}^{\ell} \mathcal{N}_{D(B)}(i_j) \right| + \left| \bigcup_{j=\ell+1}^{k} \mathcal{N}_{D(B')}(i_j) \right| - 1 \leq \left| \bigcup_{j=1}^{k} \mathcal{N}_{D(G)}(i_j) \right|.$$

Chaining these inequalities together, the $D(G)$-draconian inequality holds. A similar argument holds if $1 = i_1$, only here we explicitly write $d_{i_1} = c_1 + c'_1$ and proceed as before. In both cases the $D(G)$-draconian inequality holds, therefore $f(c, c') \in \mathfrak{D}(G)$.

Showing that f is injective is brief and straightforward, so we omit the details. What requires slightly more work is showing that f is surjective. Let $d = (d_1, \ldots, d_N) \in \mathfrak{D}(G)$. We claim that $d = f(c, c')$ where

$$c = \left(M - 1 - \sum_{i=2}^{M} d_i, d_2, \ldots, d_M \right) \quad \text{and} \quad c' = \left(N - M - \sum_{j=M+1}^{N} d_j, d_{M+1}, \ldots, d_N \right)$$

and $c \in \mathfrak{D}(B)$, $c' \in \mathfrak{D}(B')$. For notational convenience, we set

$$c_1 = M - 1 - \sum_{i=2}^{M} d_i \quad \text{and} \quad c_1' = N - M - \sum_{j=M+1}^{N} d_j.$$

Since it is clear that $d = f(c, c')$, the majority of the work will be in showing that $c \in \mathfrak{D}(B)$ and $c' \in \mathfrak{D}(B')$. The procedure is analogous for both, so we will only give the details for showing $c \in \mathfrak{D}(B)$.

By construction, the sum of entries in c is $M - 1$. Every inequality of the form

$$d_{i_1} + \cdots + d_{i_k} < \left| \bigcup_{j=1}^{k} \mathcal{N}_{D(B)}(i_j) \right| \tag{3}$$

with $1 < i_1 < \cdots < i_k \leq M$ instantly holds since the neighbors of $2, \ldots, M$ are the same in $D(G)$ and $D(B)$. It is also clear that $0 \leq c_1$ since, otherwise, $d_2 + \cdots + d_M > M - 1$, which directly contradicts (3).

Now consider a sum of a subsequence of c of the form

$$c_1 + d_{i_1} + \cdots + d_{i_k}.$$

By way of contradiction, suppose that this does not satisfy the corresponding $D(B)$-draconian inequality, that is,

$$c_1 + d_{i_1} + \cdots + d_{i_k} \geq \left| \mathcal{N}_{D(B)}(1) \cup \left(\bigcup_{j=1}^{k} \mathcal{N}_{D(B)}(i_j) \right) \right|.$$

Since $1 < i_1 < i_k \leq M$, this inequality may be rewritten

$$c_1 + d_{i_1} + \cdots + d_{i_k} \geq \left| \mathcal{N}_{D(B)}(1) \cup \left(\bigcup_{j=1}^{k} \mathcal{N}_{D(G)}(i_j) \right) \right|. \tag{4}$$

We also now know that

$$c_1' + d_{M+1} + d_{M+2} + \cdots + d_N = N - M. \tag{5}$$

Adding the corresponding sides of (4) and (5) and remembering that $c_1 + c_1' = d_1$ results in

$$d_1 + \sum_{j=1}^{k} d_{i_j} + \sum_{r=M+1}^{N} d_r \geq \left| \mathcal{N}_{D(B)}(1) \cup \left(\bigcup_{j=1}^{k} \mathcal{N}_{D(G)}(i_j) \right) \right| + N - M.$$

Using the fact that B' contains $N - M + 1$ vertices,

$$d_1 + \sum_{j=1}^{k} d_{i_j} + \sum_{r=M+1}^{N} d_r \geq \left| \mathcal{N}_{D(B)}(1) \cup \left(\bigcup_{j=1}^{k} \mathcal{N}_{D(G)}(i_j) \right) \right| + \left| \bigcup_{r=M+1}^{N} \mathcal{N}_{D(G)}(r) \right| - 1.$$

Combining the first two summands on the right side counts the vertex 1 twice, resulting in

$$d_1 + \sum_{j=1}^{k} d_{i_j} + \sum_{r=M+1}^{N} d_r \geq \left| \mathcal{N}_{D(B)}(1) \cup \left(\bigcup_{j=1}^{k} \mathcal{N}_{D(G)}(i_j) \right) \cup \left(\bigcup_{r=M+1}^{N} \mathcal{N}_{D(G)}(r) \right) \right|.$$

which is a contradiction to d being $D(G)$-draconian. Therefore the $D(B)$-draconian inequalities for c all hold, and $c \in \mathfrak{D}(B)$. An analogous argument shows $c' \in \mathfrak{D}(B')$, proving f

is a bijection. This implies $|\mathfrak{D}(B)||\mathfrak{D}(B')| = |\mathfrak{D}(G)|$; applying Theorem 2.8 completes the proof. $\square$

The next result follows quickly from induction and the recurrences proven thus far.

Corollary 3.4. If F is a forest on N vertices with k connected components then we have $\mathrm{NVol}(\nabla_F^{\mathrm{PQ}}) = 2^{N-k}$. $\square$

Interestingly, Corollary 3.4 implies that any two trees with the same number of edges will produce adjacency polytopes with the same normalized volume. This does not happen for connected graphs in general: as we will show in Example 3.11, $\mathrm{NVol}(\nabla_{C_3}^{\mathrm{PQ}}) = 6$, which is not the volume obtained from a path with three edges. Moreover, even though two trees with the same number of vertices produce adjacency polytopes with equal normalized volumes, the polytopes themselves are not combinatorially equivalent. Recall that the *f-vector* of a polytope P is the vector $(f_{-1}, f_0, \ldots, f_{\dim P})$ where f_i is the number of i-dimensional faces of P, using the convention $f_{-1} = 1$.

Example 3.5. Let G_1 and G_2 be graphs on $[4]$. Let $E(G_1) = \{12, 23, 34\}$ and $E(G_2) = \{12, 13, 14\}$. One may verify the that the f-vector of $\nabla_{G_1}^{\mathrm{PQ}}$ is

$$(1, 10, 39, 77, 82, 46, 12, 1)$$

and the f-vector of $\nabla_{G_2}^{\mathrm{PQ}}$ is

$$(1, 10, 39, 78, 86, 51, 14, 1).$$

Thus the two polytopes are not combinatorially equivalent even though Theorem 2.8 guarantees that their normalized volumes are both 8.

Through the recurrences established so far, we may reduce our work to considering only 2-connected graphs.

3.1. **The subdivision recurrence.** Given $e \in E(G)$ let $G : e$ denote the graph obtained by subdividing e. Since we are using the convention $V(G) = [N]$, we will always assume that $V(G : e) = [N + 1]$. The main result of this subsection is Theorem 3.10, which gives a recurrence for $\mathrm{NVol}(\nabla_{G:e}^{\mathrm{PQ}})$ under mild conditions. Establishing the recurrence requires multiple lemmas that have similar flavors but are distinct enough to warrant presenting their proofs.

The next three lemmas describe how to produce $D(G\,{:}\,e)$-draconian sequences from $D(G)$-draconian sequences and $D(G \setminus e)$-draconian sequences. We use the notation $A \uplus B$ to denote the disjoint union of the sets A and B.

Lemma 3.6. Let G be any connected graph on $[N]$ with an edge $e = uv$ where $\deg_G(u) = 2$. If $c \in \mathfrak{D}(G)$, then $\alpha(c) \in \mathfrak{D}(G : e)$ where $\alpha(c) = (c, 1)$. Moreover, α is an injection.

Proof. Let $c \in \mathfrak{D}(G)$. By Remark 2.9 we may assume that $e = \{N-1, N\}$ and $\deg_G(N-1) = 2$. Showing that α is an injection is routine, so we focus mainly on showing $\alpha(c) \in \mathfrak{D}(G : e)$.

Let $c = (c_1, \ldots, c_N)$. Since $c_1 + \cdots + c_N = N - 1$, the sum of entries of $\alpha(c)$ is N. By construction, $\mathcal{N}_{D(G)}(i) = \mathcal{N}_{D(G:e)}(i)$ for $i = 1, \ldots, N - 2$,

$$\mathcal{N}_{D(G:e)}(N-1) = \big(\mathcal{N}_{D(G)}(N-1) \setminus \{\overline{N}\}\big) \cup \{\overline{N+1}\}$$

and

$$\mathcal{N}_{D(G:e)}(N) = \big(\mathcal{N}_{D(G)}(N) \setminus \{\overline{N-1}\}\big) \cup \{\overline{N+1}\}.$$

Pick a sequence $1 \le i_1 < \cdots < i_k \le N + 1$. There are two cases to consider:

1. $\{\overline{N-1}, \overline{N}, \overline{N+1}\} \subsetneq \bigcup_{j=1}^{k} \mathcal{N}_{D(G:e)}(i_j)$ and
2. $\{\overline{N-1}, \overline{N}, \overline{N+1}\} \subseteq \bigcup_{j=1}^{k} \mathcal{N}_{D(G:e)}(i_j)$.

In the first case, we can deduce that $i_k \neq N+1$ and at most one of $N-1, N$ appears in $i_1, \ldots, i_k$. Therefore,

$$\left|\bigcup_{j=1}^{k} \mathcal{N}_{D(G:e)}(i_j)\right| = \left|\bigcup_{j=1}^{k} \mathcal{N}_{D(G)}(i_j)\right|$$

and

$$c_{i_1} + \cdots + c_{i_k} < \left|\bigcup_{j=1}^{k} \mathcal{N}_{D(G)}(i_j)\right| = \left|\bigcup_{j=1}^{k} \mathcal{N}_{D(G:e)}(i_j)\right|.$$

In the second case, if $i_k < N+1$, we immediately get

$$c_{i_1} + \cdots + c_{i_k} < \left|\bigcup_{j=1}^{k} \mathcal{N}_{D(G)}(i_j)\right| < \left|\bigcup_{j=1}^{k} \mathcal{N}_{D(G:e)}(i_j)\right|.$$

Otherwise, $i_k = N+1$ and

$$\left|\bigcup_{j=1}^{k} \mathcal{N}_{D(G:e)}(i_j)\right| = \left|\{\overline{N+1}\} \uplus \bigcup_{j=1}^{k-1} \mathcal{N}_{D(G)}(i_j)\right| = \left|\bigcup_{j=1}^{k-1} \mathcal{N}_{D(G)}(i_j)\right| + 1.$$

This time, we get

$$c_{i_1} + \cdots + c_{i_k} = c_{i_1} + \cdots + c_{i_{k-1}} + 1 < \left|\bigcup_{j=1}^{k-1} \mathcal{N}_{D(G)}(i_j)\right| + 1 = \left|\bigcup_{j=1}^{k} \mathcal{N}_{D(G:e)}(i_j)\right|.$$

Since each case results in satisfying the $D(G:e)$-draconian inequalities, we have shown that $\alpha(c) \in \mathfrak{D}(G:e)$. □

Lemma 3.7. Let G be a 2-connected graph and let $e = uv$ be any edge. If $c \in \mathfrak{D}(G \setminus e)$, then $\beta(c) \in \mathfrak{D}(G:e)$ where $\beta(c) = \alpha(c) + e_u - e_{N+1}$. Moreover, β is an injection.

Proof. Arguing that β is injective is routine, so its details are omitted. For what remains, by Remark 2.9 we may assume that $e = \{N-1, N\}$. We then want to show that, if $c = (c_1, \ldots, c_N) \in \mathfrak{D}(G \setminus e)$, then

$$\beta(c) = (c_1, \ldots, c_{N-2}, c_{N-1} + 1, c_N, 0) \in \mathfrak{D}(G:e).$$

Set $\beta(c) = (\beta_1, \ldots, \beta_{N+1})$. Let $1 \leq i_1 < \cdots < i_k \leq N+1$ and set $\ell = k$ if $i_k < N+1$ and $\ell = k-1$ if $i_k = N+1$. If $N-1 \neq i_j$ for any j, then

$$\beta_{i_1} + \cdots + \beta_{i_k} = c_{i_1} + \cdots + c_{i_\ell} < \left|\bigcup_{j=1}^{\ell} \mathcal{N}_{D(G\setminus e)}(i_j)\right| \leq \left|\bigcup_{j=1}^{k} \mathcal{N}_{D(G:e)}(i_j)\right|.$$

Otherwise, $N-1 = i_j$ for some j. In this case,

$$\left|\bigcup_{j=1}^{\ell} \mathcal{N}_{D(G\setminus e)}(i_j)\right| = \left|\bigcup_{j=1}^{\ell} \mathcal{N}_{D(G:e)}(i_j)\right| - 1 \leq \left|\bigcup_{j=1}^{k} \mathcal{N}_{D(G:e)}(i_j)\right| - 1.$$

Together we have

$$\begin{aligned}\beta_{i_1}+\cdots+\beta_{i_k} &= c_{i_1}+\cdots+c_{i_k}+1\\ &< \left|\bigcup_{j=1}^{\ell}\mathcal{N}_{D(G\setminus e)}(i_j)\right|+1\\ &\le \left|\bigcup_{j=1}^{k}\mathcal{N}_{D(G:e)}(i_j)\right|,\end{aligned}$$

and the $D(G:e)$-draconian inequality holds. Therefore $\beta(c)\in\mathfrak{D}(G:e)$. $\square$

Lemma 3.8. Let G be a 2-connected graph with an edge $e=uv$ where $\deg_G(u)=2$. If $c\in\mathfrak{D}(G)$, then $\gamma(c)\in\mathfrak{D}(G:e)$ where $\gamma(c)$ is formed by the following rule. Set $\gamma'(c)=\alpha(c)-e_u+e_{N+1}$.

1. If $c\notin\mathfrak{D}(G\setminus e)$ then
   (a) If $\gamma'(c)\in\mathfrak{D}(G:e)$ then set $\gamma(c)=\gamma'(c)$.
   (b) If $\gamma'(c)\notin\mathfrak{D}(G:e)$ then set $\gamma(c)=\alpha(c)+e_u-e_{N+1}$.
2. if $c\in\mathfrak{D}(G\setminus e)$, then
   (a) If $\gamma'(c)\in\mathfrak{D}(G:e)$ then set $\gamma(c)=\gamma'(c)$.
   (b) If $\gamma'(c)\notin\mathfrak{D}(G:e)$ then set $\gamma(c)=\alpha(c)+e_v-e_{N+1}$.

Additionally, γ is an injection.

Proof. As usual, Remark 2.9 allows us to assume $e=\{N-1,N\}$ and $\deg_G(N-1)=2$. This allows us to more specifically rewrite γ as follows: set $\gamma'(c)=(c_1,\dots,c_{N-2},c_{N-1}-1,c_N,2)$.

1. If $c\notin\mathfrak{D}(G\setminus e)$ then
   (a) If $\gamma'(c)\in\mathfrak{D}(G:e)$ then set $\gamma(c)=\gamma'(c)$.
   (b) If $\gamma'(c)\notin\mathfrak{D}(G:e)$ then set $\gamma(c)=(c_1,\dots,c_{N-2},c_{N-1}+1,c_N,0)$.
2. if $c\in\mathfrak{D}(G\setminus e)$, then
   (a) If $\gamma'(c)\in\mathfrak{D}(G:e)$ then set $\gamma(c)=\gamma'(c)$.
   (b) If $\gamma'(c)\notin\mathfrak{D}(G:e)$ then set $\gamma(c)=(c_1,\dots,c_{N-1},c_N+1,0)$.

Throughout the proof we use the notation $\gamma(c)=(\gamma_1,\dots,\gamma_{N+1})$.

First suppose $c\notin\mathfrak{D}(G\setminus e)$ and $\gamma'(c)\notin\mathfrak{D}(G{:}e)$, so that $\gamma(c)=(c_1,\dots,c_{N-2},c_{N-1}+1,c_N,0)$. We want to show this sequence is $D(G:e)$-draconian. Assume to the contrary that $\gamma(c)\notin\mathfrak{D}(G:e)$. There must then be a sequence $1\le i_1<\cdots<i_k\le N+1$ for which

$$\gamma_{i_1}+\cdots+\gamma_{i_k}\ge\left|\bigcup_{j=1}^{k}\mathcal{N}_{D(G:e)}(i_j)\right|. \tag{6}$$

Because $c\in\mathfrak{D}(G)$ and $\mathcal{N}_{D(G)}(i_j)\subseteq\mathcal{N}_{D(G:e)}(i_j)$ for all i_j, (6) could only hold if $i_j=N-1$ for some j. If $i_k=N$ then set $\ell=k$ and if $i_k=N+1$ then set $\ell=k-1$. For these cases we have

$$\bigcup_{j=1}^{k}\mathcal{N}_{D(G:e)}(i_j)=\{\overline{N+1}\}\uplus\bigcup_{j=1}^{\ell}\mathcal{N}_{D(G)}(i_j).$$

Putting this into (6), we get

$$c_{i_1}+\cdots+c_{i_\ell}+1=\gamma_{i_1}+\cdots+\gamma_{i_k}\ge 1+\left|\bigcup_{j=1}^{\ell}\mathcal{N}_{D(G)}(i_j)\right|,$$

contradicting $c \in \mathfrak{D}(G)$. Thus, $i_k = N-1$.

Since $c \in \mathfrak{D}(G)$ and $\deg_G(N-1) = 2$, we know $c_{N-1} \in \{0,1,2\}$. Therefore, $\gamma_{N-1} \in \{1,2,3\}$. If $\gamma_{N-1} = 3$ then $\gamma'(c) = (c_1, \dots, c_{N-2}, 1, c_N, 2)$. However, one may show directly that this sequence is $D(G:e)$-draconian, a contradiction. A similar issue occurs if $\gamma_{N-1} = 2$, since, this time, $\gamma'_{N-1} = 0$. Therefore, $\gamma_{N-1} = 1$. By applying an argument similar to that in the proof of Lemma 3.6, one may again show directly that $\gamma'(c)$ is $D(G:e)$-draconian, a contradiction. Thus, $\gamma(c) \in \mathfrak{D}(G:e)$.

Now suppose that $c \in \mathfrak{D}(G\backslash e)$ and $\gamma'(c) \notin \mathfrak{D}(G{:}e)$. Proving that $\gamma(c) = (c_1, \dots, c_{N-1}, c_N, 0)$ follows the proof of Lemma 3.7 almost identically, replacing $N-1$ with N. For this reason, we omit the details.

To show that γ is an injection, we can restrict to comparing the sequences of 1(a) with those of 2(a) and the sequences of 1(b) with those of 2(b). Fortunately, it is straightforward to see that no sequence can arise simultaneously as $\gamma(c)$ under the conditions of 1(a) and $\gamma(c')$ under the conditions of 2(a). If this were possible, we would obtain $c = c'$, but c cannot simultaneously be a member of and absent from $\mathfrak{D}(G \setminus e)$.

For the remaining case, suppose $\gamma(c) = \gamma(c')$ where $\gamma(c)$ falls under the conditions of 1(b) and $\gamma(c')$ falls under the conditions of 2(b). Let $\gamma(c') = (\gamma'_1, \dots, \gamma'_{N+1})$. As we saw previously in this proof, in order to have $\gamma(c') \in \mathfrak{D}(G:e)$ we need $\gamma'_{N-1} = 0$ since all other possibilities resulted in a contradiction. However, in $\gamma(c)$, the same coordinate is $c_{N-1} + 1 > 0$. Thus the two cannot be equal, causing a contradiction, and γ is injective. □

Fix a particular edge e of a 2-connected graph G for which one of the endpoints has degree 2 in G. Let $\mathcal{A}_G(e)$, $\mathcal{B}_G(e)$, and $\mathcal{C}_G(e)$ be the set of $D(G:e)$-draconian sequences constructed from α, β, and γ in Lemmas 3.6, 3.7, and 3.8, respectively.

Lemma 3.9. Let G be a 2-connected graph with an edge $e = uv$ such that $\deg_G(u) = 2$. The sets $\mathcal{A}_G(e)$, $\mathcal{B}_G(e)$, and $\mathcal{C}_G(e)$ are pairwise disjoint.

Proof. We continue to use the convention that $e = \{N-1, N\}$ and $\deg_G(N-1) = 2$. By comparing the values of c_{N+1}, it is clear that $\mathcal{A}_G(e) \cap \mathcal{B}_G(e) = \emptyset$ and $\mathcal{A}_G(e) \cap \mathcal{C}_G(e) = \emptyset$. Thus we only need to focus on $\mathcal{B}_G(e) \cap \mathcal{C}_G(e)$. In fact, since γ is an injection, we only need to consider elements of $\mathcal{C}_G(e)$ that fall under the conditions of 1(b) or 2(b) of Lemma 3.8.

Suppose that $\beta(c) = \gamma(c')$ and write

$$c = (c_1, \dots, c_{N+1}) \text{ and } c' = (c'_1, \dots, c'_{N+1}).$$

If $\gamma(c)$ satisfies the conditions of 1(b) in Lemma 3.8, it follows that $c = c'$, which is impossible since this this requires $c \notin \mathfrak{D}(G \setminus e)$ and $c' \in \mathfrak{D}(G \setminus e)$. Thus assume $c \in \mathfrak{D}(G \setminus e)$. Since both $c, c' \in \mathfrak{D}(G \setminus e)$, we know $c_{N-1}, c'_{N-1} \le 1$. By the definitions of β and γ, we have $c'_{N-1} = c_{N-1} + 1$ and $c_{N-1} = c'_{N-1} + 1$, so this forces both c_{N-1} and c'_{N-1} to be positive, meaning $c_{N-1} = c'_{N-1} = 1$. This again implies $c = c'$, which contradicts $\beta(c) = \gamma(c') = \gamma(c)$. Therefore, $\mathcal{B}_G(e) \cap \mathcal{C}_G(e) = \emptyset$, completing the proof. □

This result, together with the three lemmas preceding it, give $\mathcal{A}_G(e) \uplus \mathcal{B}_G(e) \uplus \mathcal{C}_G(e) \subseteq \mathfrak{D}(G{:}e)$. It turns out that the reverse inclusion holds, establishing what we call *the subdivision recurrence.*

Theorem 3.10 (Subdivision recurrence)**.** Let G be a 2-connected graph with an edge $e = uv$ where $\deg_G(u) = 2$. Then $\mathfrak{D}(G:e) = \mathcal{A}_G(e) \uplus \mathcal{B}_G(e) \uplus \mathcal{C}_G(e)$ and, consequently,

$$\operatorname{NVol}(\nabla^{\mathrm{PQ}}_{G:e}) = 2 \operatorname{NVol}(\nabla^{\mathrm{PQ}}_{G}) + \operatorname{NVol}(\nabla^{\mathrm{PQ}}_{G\setminus e}).$$

Proof. Again without loss of generality we may assume $e = \{N-1, N\}$ and $\deg_G(N-1) = 2$. By Lemmas 3.6, 3.7, and 3.8,

$$\mathcal{A}_G(e) \cup \mathcal{B}_G(e) \cup \mathcal{C}_G(e) \subseteq \mathfrak{D}(G:e).$$

For the reverse inclusion, we will show that, given $d = (d_1, \dots, d_{N+1}) \in \mathfrak{D}(G:e)$, one of the following conditions holds:

1. If $d_{N+1} = 2$ then $(d_1, \dots, d_{N-2}, d_{N-1}+1, d_N) \in \mathfrak{D}(G)$.
2. If $d_{N+1} = 1$ then $(d_1, \dots, d_N) \in \mathfrak{D}(G)$.
3. If $d_{N+1} = 0$ then one of the following is true:
   (a) $(d_1, \dots, d_{N-2}, d_{N-1}-1, d_N) \in \mathfrak{D}(G \setminus e)$;
   (b) both $(d_1, \dots, d_{N-2}, d_{N-1}-2, d_N, 2) \notin \mathfrak{D}(G:e)$ and $(d_1, \dots, d_{N-2}, d_{N-1}-1, d_N) \in \mathfrak{D}(G) \setminus \mathfrak{D}(G \setminus e)$; or
   (c) both $(d_1, \dots, d_{N-2}, d_{N-1}-1, d_N-1, 2) \notin \mathfrak{D}(G:e)$ and $(d_1, \dots, d_{N-2}, d_{N-1}, d_N-1) \in \mathfrak{D}(G \setminus e)$.

If the second condition holds, then $d \in \mathcal{A}_G(e)$; if condition 3(a) holds, then $d \in \mathcal{B}_G(e)$; if any of the remaining conditions hold, then $d \in \mathcal{C}_G(e)$.

First suppose $d_{N+1} = 2$ and let $1 \le i_1 < \cdots < i_k \le N$. Set $(c_1, \dots, c_N) = (d_1, \dots, d_{N-2}, d_{N-1}+1, d_N)$. If $i_k < N-1$ then $\mathcal{N}_{D(G:e)}(i_j) = \mathcal{N}_{D(G)}(i_j)$ for each j, so the corresponding draconian inequality

$$c_{i_1} + \cdots + c_{i_k} = d_{i_1} + \cdots + d_{i_k} < \left| \bigcup_{j=1}^{k} \mathcal{N}_{D(G:e)}(i_j) \right| = \left| \bigcup_{j=1}^{k} \mathcal{N}_{D(G)}(i_j) \right|$$

holds. Otherwise, since $d \in \mathfrak{D}(G:e)$,

$$\begin{aligned} c_{i_1} + \cdots + c_{i_k} &\le d_{i_1} + \cdots + d_{i_k} + 2 - 1 \\ &< \left| \left(\bigcup_{j=1}^{k} \mathcal{N}_{D(G:e)}(i_j) \right) \cup \mathcal{N}_{D(G:e)}(N+1) \right| - 1 \\ &= \left| \left(\bigcup_{j=1}^{k} \mathcal{N}_{D(G)}(i_j) \right) \uplus \{\overline{N+1}\} \right| - 1 \\ &= \left| \bigcup_{j=1}^{k} \mathcal{N}_{D(G)}(i_j) \right|. \end{aligned}$$

Therefore, each $D(G)$-draconian inequality holds for $(c_1, \dots, c_N)$, establishing the first condition.

Next suppose $d_{N+1} = 1$ and let $1 \le i_1 < \cdots < i_k \le N$. If $i_k < N-1$ then the corresponding draconian inequality holds as in the case of $d_{N+1} = 2$. If $i_k \ge N-1$ then we know from $d \in \mathfrak{D}(G:e)$ that

$$\begin{aligned} d_{i_1} + \cdots + d_{i_k} + 1 &< \left| \left(\bigcup_{j=1}^{k} \mathcal{N}_{D(G:e)}(i_j) \right) \cup \mathcal{N}_{D(G:e)}(N+1) \right| \\ &= \left| \left(\bigcup_{j=1}^{k} \mathcal{N}_{D(G)}(i_j) \right) \uplus \{\overline{N+1}\} \right| \\ &= \left| \bigcup_{j=1}^{k} \mathcal{N}_{D(G)}(i_j) \right| + 1. \end{aligned}$$

Subtracting 1 from both sides establishes the corresponding $D(G)$-draconian inequality for $(d_1, \ldots, d_N)$. Thus the second condition holds.

Establishing the last condition, where $d_{N+1} = 0$, requires the most care. Since $\deg_G(N-1) = 2$, we know that $d_{N-1} \in \{0, 1, 2\}$ and we will treat each case separately.

Suppose $d_{N-1} = 0$. Our aim will be to show that condition 3(c) holds. It is clear that $(d_1, \ldots, d_{N-2}, d_{N-1} - 1, d_N - 1, 2) \notin \mathfrak{D}(G:e)$ since $d_{N-1} - 1 < 0$. Now, if $d_N = 0$ then there is a contradiction, since this implies

$$N = d_1 + \cdots + d_{N-2} < \left| \bigcup_{j=1}^{N-2} \mathcal{N}_{D(G:e)}(j) \right| = \left| \bigcup_{j=1}^{N-2} \mathcal{N}_{D(G)}(j) \right| = N.$$

Thus, $d_N > 0$.

Set $(c_1, \ldots, c_N) = (d_1, \ldots, d_{N-1}, d_N - 1)$. If $c_{i_1} + \cdots + c_{i_k}$ with $i_k < N-1$, then the desired $D(G)$-draconian inequality holds using the same argument as for the previous conditions. If $i_k = N-1$ then

$$c_{i_1} + \cdots + c_{i_k} = d_{i_1} + \cdots + d_{i_{k-1}} < \left| \bigcup_{j=1}^{k-1} \mathcal{N}_{D(G:e)}(i_j) \right| = \left| \bigcup_{j=1}^{k-1} \mathcal{N}_{D(G \setminus e)}(i_j) \right| \le \left| \bigcup_{j=1}^{k} \mathcal{N}_{D(G \setminus e)}(i_j) \right|.$$

Lastly, if $i_k = N$, then

$$\begin{aligned} c_{i_1} + \cdots + c_{i_k} &= d_{i_1} + \cdots + d_{i_k} - 1 \\ &< \left| \bigcup_{j=1}^{k} \mathcal{N}_{D(G:e)}(i_j) \right| - 1 \\ &= \left| \left(\bigcup_{j=1}^{k} \mathcal{N}_{D(G \setminus e)}(i_j) \right) \uplus \{\overline{N+1}\} \right| - 1 \\ &= \left| \bigcup_{j=1}^{k} \mathcal{N}_{D(G \setminus e)}(i_j) \right|. \end{aligned}$$

Therefore, if $d_{N-1} = 0$ then condition 3(c) holds.

Next suppose $d_{N-1} = 2$. Condition 3(c) clearly cannot hold since this condition requires $d_{N-1} \le 1$, so we must show that either 3(a) or 3(b) holds. Suppose that condition 3(a) does not hold, that is, suppose $(d_1, \ldots, d_{N-2}, 1, d_N) \notin \mathfrak{D}(G \setminus e)$. Showing that this sequence is in $\mathfrak{D}(G)$ can be done directly repeating our by-now-usual strategies, so the sequence is in $\mathfrak{D}(G) \setminus \mathfrak{D}(G \setminus e)$.

To show that $(d_1,\dots,d_{N-2},0,d_N,2)\notin\mathfrak{D}(G\,{:}\,e)$, observe that $(d_1,\dots,d_{N-2},1,d_N)\notin\mathfrak{D}(G\setminus e)$ implies there is some inequality of the form

$$d_{i_1}+\cdots+d_{i_k}\geq\left|\bigcup_{j=1}^{k}\mathcal{N}_{D(G\setminus e)}(i_j)\right| \tag{7}$$

with $i_k=N$ and $i_{k-1}<N-1$. If $N-1\notin\bigcup_{j=1}^{k}\mathcal{N}_{D(G\setminus e)}(i_j)$ then add 2 to both sides of (7) to get

$$\begin{aligned}d_{i_1}+\cdots+d_{i_k}+2&\geq\left|\bigcup_{j=1}^{k}\mathcal{N}_{D(G\setminus e)}(i_j)\right|+2\\&=\left|\bigcup_{j=1}^{k}\mathcal{N}_{D(G\setminus e)}(i_j)\right|+\left|\{\overline{N-1},\overline{N+1}\}\right|\\&=\left|\bigcup_{j=1}^{k}\mathcal{N}_{D(G:e)}(i_j)\right|,\end{aligned}$$

which would imply $(d_1,\dots,d_{N-2},0,d_N,2)\notin\mathfrak{D}(G\,{:}\,e)$. If $N-1\in\left|\bigcup_{j=1}^{k}\mathcal{N}_{D(G\setminus e)}(i_j)\right|$ then add 2 to the left side of (7) and $1=\left|\{\overline{N}\}\right|$ to the right side; the conclusion is the same. Thus, if condition 3(a) does not hold then condition 3(b) does hold.

For the case of when $d_{N-1}=1$, the first part of condition 3(b) clearly holds. Verifying that $(d_1,\dots,d_{N-2},0,d_N)\in\mathfrak{D}(G)$ is now routine, so either condition 3(a) holds or 3(b) holds.

We have shown that, regardless of value of d_{N+1}, one of the three conditions holds, hence $d\in\mathcal{A}_G(e)\cup\mathcal{B}_G(e)\cup\mathcal{C}_G(e)$ and $\mathfrak{D}(G\,{:}\,e)=\mathcal{A}_G(e)\cup\mathcal{B}_G(e)\cup\mathcal{C}_G(e)$. By Lemma 3.9, this union is disjoint, so

$$\begin{aligned}|\mathfrak{D}(G:e)|&=|\mathcal{A}_G(e)\uplus\mathcal{B}_G(e)\uplus\mathcal{C}_G(e)|\\&=|\mathcal{A}_G(e)|+|\mathcal{B}_G(e)|+|\mathcal{C}_G(e)|\\&=2|\mathfrak{D}(G)|+|\mathfrak{D}(G\setminus e)|.\end{aligned}$$

Applying Theorem 2.8, the result is proven. □

Example 3.11. Consider $C_3=([3],\{12,13,23\})$ and let $e=13$; there are six $D(C_3)$-draconian sequences:

$$\begin{array}{ccc}(2,0,0)&(0,2,0)&(0,0,2)\\(1,1,0)&(1,0,1)&(0,1,1)\end{array}$$

Subdividing e replaces 13 with edges 34 and 14 to obtain C_4. By the subdivision recurrence, $\mathfrak{D}(C_4)=\mathfrak{D}(C_3:e)=\mathcal{A}_{C_3}(e)\uplus\mathcal{B}_{C_3}(e)\uplus\mathcal{C}_{C_3}(e)$. Following the definitions of α, β, and γ we obtain

$$\begin{aligned}\mathcal{A}_{C_3}(e)&=\{(2,0,0,1),(0,2,0,1),(0,0,2,1),(1,1,0,1),(1,0,1,1),(0,1,1,1)\}\\\mathcal{B}_{C_3}(e)&=\{(1,2,0,0),(2,1,0,0),(2,0,1,0),(1,1,1,0)\}\\\mathcal{C}_{C_3}(e)&=\{(1,0,0,2),(1,0,2,0),(0,1,0,2),(0,0,1,2),(0,1,2,0),(0,2,1,0)\}.\end{aligned}$$

Notice that $|\mathfrak{D}(C_3)|=3\cdot 2^1$ and $|\mathfrak{D}(C_4)|=4\cdot 2^2$.

Example 3.11 can be easily generalized to all cycles through induction, leading to our first formula for a 2-connected graph.

Corollary 3.12. For every N-cycle C_N, $\mathrm{NVol}(\nabla^{\mathrm{PQ}}_{C_N})=N2^{N-2}$. □

The subdivision recurrence does not necessarily hold if we allow both endpoints of e to have degree larger than 2 in G. For example, if $G = K_1 \vee P_3$, where P_3 is the path on three vertices, and e is the edge of G whose endpoints each have degree 3 in G, then one may show that $\mathrm{NVol}(\nabla^{\mathrm{PQ}}_{G:e}) = 50$ whereas

$$2\,\mathrm{NVol}(\nabla^{\mathrm{PQ}}_{G:e}) + \mathrm{NVol}(\nabla^{\mathrm{PQ}}_{G\setminus e}) = 2(18) + 16 = 52.$$

Question 3.13. Under what conditions for a graph G and an edge e is there a “nice” recurrence for $\mathrm{NVol}(\nabla^{\mathrm{PQ}}_{G:e})$?

3.2. **The triangle recurrence.** The framework which establishes the subdivision recurrence can be adapted to a different operation. Given an edge $e = uv$ of a graph G, let $G\triangle e$ denote the graph with edge set $E(G) \cup \{uw, vw\}$ where w is a new vertex. We will continue to assume $V(G) = [N]$ and $V(G\triangle e) = [N + 1]$. As in Section 3.1, establishing a recurrence formula for $\mathfrak{D}(G\triangle e)$ will require establishing several smaller results first. The first two of these have proofs analogous enough to the proofs of Lemma 3.6 and Lemma 3.7, respectively, that we omit their details.

Lemma 3.14. Let G be any connected graph on $[N]$ and e any edge. If $c \in \mathfrak{D}(G)$, then $\alpha^{\triangle}(c) \in \mathfrak{D}(G\triangle e)$ where $\alpha^{\triangle}(c) = (c, 1)$. Moreover, $\alpha^{\triangle}$ is injective. □

Lemma 3.15. Let G be a connected graph on $[N]$ and let $e = uv$ be an edge with $\deg_G(u) = 2$. If $c \in \mathfrak{D}(G)$, then $\beta^{\triangle}(c) \in \mathfrak{D}(G\triangle e)$ where

$$\beta^{\triangle}(c) = \alpha^{\triangle}(c) + e_u - e_{N+1}.$$

Additionally, $\beta^{\triangle}$ is injective. □

The next lemma is analogous to Lemmas 3.8, but this time its proof is different enough for us to justify providing it. It will be helpful to introduce the analogues of $\mathcal{A}_G(e)$ and $\mathcal{B}_G(e)$ here: let $\mathcal{A}^{\triangle}_G(e)$ and $\mathcal{B}^{\triangle}_G(e)$ be the $D(G\triangle e)$-draconian sequences constructed with $\alpha^{\triangle}$ and $\beta^{\triangle}$, and $\gamma^{\triangle}$ in Lemmas 3.14 and 3.15, respectively.

Lemma 3.16. Let G be a connected graph on $[N]$ and let $e = uv$ be an edge for which $\deg_G(u) = 2$. If $c \in \mathfrak{D}(G)$, then $\gamma^{\triangle}(c) \in \mathfrak{D}(G\triangle e)$ where

$$\gamma^{\triangle}(c) = \begin{cases} \alpha^{\triangle}(c) + e_v - e_{N+1} & \text{if not in } \mathcal{B}^{\triangle}_G(e) \\ \alpha^{\triangle}(c) - e_u + e_{N+1} & \text{otherwise.} \end{cases}$$

Additionally, $\gamma^{\triangle}$ is injective.

Proof. That $\gamma^{\triangle}$ is injective is clear. For what remains, by Remark 2.9 we again assume without loss of generality that $e = \{N-1, N\}$ and $\deg_G(N-1) = 2$. So, if $c = (c_1, \ldots, c_N) \in \mathfrak{D}(G)$, then we must prove $\gamma^{\triangle}(c) \in \mathfrak{D}(G\triangle e)$ where

$$\gamma^{\triangle}(c) = \begin{cases} (c_1, \ldots, c_{N-2}, c_{N-1}, c_N + 1, 0) & \text{if not in } \mathcal{B}^{\triangle}_G(e) \\ (c_1, \ldots, c_{N-2}, c_{N-1} - 1, c_N, 2) & \text{otherwise.} \end{cases}$$

Note that, in both cases, the entries sum to N.

If $\gamma^{\triangle}(c) = (c_1, \ldots, c_{N-2}, c_{N-1}, c_N + 1, 0)$ then showing it is $D(G\triangle e)$-draconian is entirely analogous to the proof of Lemma 3.15. Otherwise, $\gamma^{\triangle}(c) = (c_1, \ldots, c_{N-2}, c_{N-1} - 1, c_N, 2)$. Since we are in this case, we know $c_{N-1} \geq 1$, so all entries of $\gamma^{\triangle}(c)$ are nonnegative.

Let $\gamma^\triangle(c) = (\gamma^\triangle_1, \ldots, \gamma^\triangle_{N+1})$ and $1 \le i_1 < \cdots < i_k \le N+1$. If $i_k < N+1$ then the corresponding $D(G\triangle e)$-draconian inequality holds due to $c \in \mathfrak{D}(G)$, to $\gamma_{i_j} \le c_{i_j}$ for each j, and to $\mathcal{N}_G(i_j) \subseteq \mathcal{N}_{G\triangle e}(i_j)$ for each j. If $i_k = N+1$ and $i_j = N$ for some j, then

$$\gamma^\triangle_{i_1} + \cdots + \gamma^\triangle_{i_k} = \gamma^\triangle_{i_1} + \cdots + \gamma^\triangle_{i_{k-1}} + 1 < \left| \bigcup_{j=1}^{k-1} \mathcal{N}_{D(G)}(i_j) \right| + |\{\overline{N+1}\}| = \left| \bigcup_{j=1}^{k} \mathcal{N}_{D(G\triangle e)}(i_j) \right|.$$

Lastly, suppose $i_k = N+1$ and $i_j \neq N-1$ for all j. If $N-1$ is not a neighbor of i_j for any $j < k$, then

$$\gamma^\triangle_{i_1} + \cdots + \gamma^\triangle_{i_k} = \gamma^\triangle_{i_1} + \cdots + \gamma^\triangle_{i_k} < \left| \bigcup_{j=1}^{k-1} \mathcal{N}_{D(G)}(i_j) \right| + |\{\overline{N-1}, \overline{N+1}\}| \le \left| \bigcup_{j=1}^{k} \mathcal{N}_{D(G\triangle e)}(i_j) \right|.$$

Otherwise, we know that

$$\bigcup_{j=1}^{k} \mathcal{N}_{D(G\triangle e)}(i_j) = \mathcal{N}_{D(G\triangle e)}(N-1) \cup \bigcup_{j=1}^{k} \mathcal{N}_{D(G\triangle e)}(i_j).$$

Since $c_{N-1} \le \deg_G(N-1) = 2$, we know $c_{N-1} = 0$ or $c_{N-1} = 1$.

If $c_{N-1} = 0$ then

$$\begin{aligned}
\gamma^\triangle_{i_1} + \cdots + \gamma^\triangle_{i_k} &= c_{i_1} + \cdots + c_{i_{k-1}} + c_{N-1} + 1 \\
&< \left| \mathcal{N}_{D(G\triangle e)}(N-1) \cup \bigcup_{j=1}^{k} \mathcal{N}_{D(G\triangle e)}(i_j) \right| \\
&= \left| \bigcup_{j=1}^{k} \mathcal{N}_{D(G\triangle e)}(i_j) \right|
\end{aligned}$$

where the first inequality follows from having proven $\alpha^\triangle(c) \in \mathfrak{D}(G\triangle e)$ in Lemma 3.14. If $c_{N-1} = 1$ then

$$\begin{aligned}
\gamma^\triangle_{i_1} + \cdots + \gamma^\triangle_{i_k} &= c_{i_1} + \cdots + c_{i_{k-1}} + c_{N-1} \\
&< \left| \mathcal{N}_{D(G\triangle e)}(N-1) \cup \bigcup_{j=1}^{k} \mathcal{N}_{D(G\triangle e)}(i_j) \right| \\
&= \left| \bigcup_{j=1}^{k} \mathcal{N}_{D(G\triangle e)}(i_j) \right|.
\end{aligned}$$

In all cases, we have shown $\gamma^\triangle(c) \in \mathfrak{D}(G\triangle e)$, which completes the proof. □

Let $\mathcal{C}^\triangle_G(e)$ be the $D(G\triangle e)$-draconian sequences constructed from $\gamma^\triangle$ in Lemma 3.16. The proof of the following is completely analogous to the proof of Lemma 3.9.

Lemma 3.17. Let G be a graph having an edge $e = uv$ with $\deg_G(u) = 2$. The sets $\mathcal{A}^\triangle_G(e)$, $\mathcal{B}^\triangle_G(e)$, and $\mathcal{C}^\triangle_G(e)$ are pairwise disjoint. □

As in Section 3.1, the previous four lemmas imply $\mathcal{A}^\triangle_G(e) \uplus \mathcal{B}^\triangle_G(e) \uplus \mathcal{C}^\triangle_G(e) \subseteq \mathfrak{D}(G\triangle e)$. The reverse inclusion again holds, establishing what we call *the triangle recurrence.*

Theorem 3.18 (Triangle Recurrence)**.** Let G be any connected graph on $[N]$ for which $e = uv$ with $\deg_G(u) = 2$. Then

$$\mathrm{NVol}(\nabla^{\mathrm{PQ}}_{G\triangle e}) = 3\,\mathrm{NVol}(\nabla^{\mathrm{PQ}}_{G}).$$

Proof. As usual we assume $V(G) = [N]$, $e = \{N-1, N\}$, and $\deg_G(N-1) = 2$. Lemmas 3.14, 3.15, and 3.16 show that

$$\mathcal{A}^{\triangle}_G(e) \cup \mathcal{B}^{\triangle}_G(e) \cup \mathcal{C}^{\triangle}_G(e) \subseteq \mathfrak{D}(G\triangle e),$$

so we must show the reverse inclusion holds. Let $d = (d_1, \dots, d_{N+1}) \in \mathfrak{D}(G\triangle e)$. As with the subdivision recurrence, there are three statements we must show:

1. If $d_{N+1} = 1$, then $(d_1, \dots, d_N) \in \mathfrak{D}(G)$;
2. If $d_{N+1} = 0$, then one of $(d_1, \dots, d_{N-2}, d_{N-1} - 1, d_N)$ and $(d_1, \dots, d_{N-2}, d_{N-1}, d_N - 1)$ is in $\mathfrak{D}(G)$; and
3. If $d_{N+1} = 2$ then both $(d_1, \dots, d_{N-2}, d_{N-1}+1, d_N+1) \in \mathcal{B}^{\triangle}_G(e)$ and $(d_1, \dots, d_{N-2}, d_{N-1}+1, d_N) \in \mathfrak{D}(G)$.

First suppose $d_{N+1} = 1$ and pick any $D(G\triangle e)$-draconian sequence of the form $(d_1, \dots, d_N, 1)$. Let $1 \le i_1 < \cdots < i_k \le N$. If $i_j \ne N-1, N$ for all j then the neighbors of i_j are the same in $D(G\triangle e)$ and $D(G)$, so the corresponding $D(G)$-draconian inequality instantly holds. Otherwise,

$$\begin{aligned} d_{i_1} + \cdots + d_{i_k} &= d_{i_1} + \cdots + d_{i_k} + 1 - 1 \\ &< \left| \left(\bigcup_{j=1}^{k} \mathcal{N}_{D(G\triangle e)}(i_j) \right) \cup \mathcal{N}_{D(G\triangle e)}(N+1) \right| - 1 \\ &= \left| \left(\bigcup_{j=1}^{k} \mathcal{N}_{D(G\triangle e)}(i_j) \right) \cup \{\overline{N+1}\} \right| - 1 \\ &= \left| \bigcup_{j=1}^{k} \mathcal{N}_{D(G)}(i_j) \right| \end{aligned}$$

Thus $(d_1, \dots, d_N) \in \mathfrak{D}(G)$.

Next, suppose $d_{N+1} = 0$. Since d_{N-1}, d_N, d_{N+1} cannot all be 0, we know $d_{N-1} > 0$ or $d_N > 0$. Suppose first that $d_N = 0$, in which case $d_{N-1} > 0$. Set

$$d' = (d'_1, \dots, d'_N) = (d_1, \dots, d_{N-2}, d_{N-1} - 1, d_N)$$

and

$$d'' = (d''_1, \dots, d''_N) = (d_1, \dots, d_{N-2}, d_{N-1}, d_N - 1)$$

and consider a sequence $1 \le i_1 < \cdots < i_k \le N$. If $i_k < N-1$ then the corresponding $D(G)$-draconian inequality for d' holds since the neighbors of i_j are the same in $D(G\triangle e)$ and $D(G)$. If $i_k = N-1$ then

$$\begin{aligned} d'_{i_1} + \cdots + d'_{i_k} &= d_{i_1} + \cdots + d_{i_k} - 1 \\ &< \left| \bigcup_{j=1}^{k} \mathcal{N}_{D(G\triangle e)}(i_j) \right| - 1 \\ &= \left| \bigcup_{j=1}^{k} \mathcal{N}_{D(G)}(i_j) \right| \end{aligned}$$

Lastly consider when $i_k = N$. Since we are currently assuming $d_N = 0$, we use our previous cases to directly show

$$d'_{i_1} + \cdots + d'_{i_k} = d'_{i_1} + \cdots + d'_{i_{k-1}} < \left| \bigcup_{j=1}^{k-1} \mathcal{N}_{D(G)}(i_j) \right| \leq \left| \bigcup_{j=1}^{k} \mathcal{N}_{D(G)}(i_j) \right|.$$

Under these assumptions, $d' \in \mathfrak{D}(G)$. The analogous argument holds if $d_{N-1} = 0$, in which case $d'' \in \mathfrak{D}(G)$.

We still must consider when both $d_{N-1}, d_N \geq 1$. Consider d' and select a sequence $1 \leq i_1 < \cdots < i_k \leq N$. If $i_k < N$ or if $i_k = N$ and $i_{k-1} = N-1$ then the corresponding $D(G)$-draconian inequalities hold for d' as before. For the case of $i_k = N$ and $i_{k-1} < N-1$, notice that for any such sequence,

$$d_{i_1} + \cdots + d_{i_k} < \left| \bigcup_{j=1}^{k} \mathcal{N}_{D(G\triangle e)}(i_j) \right| - 1. \tag{8}$$

This is because, when comparing the above inequality to the inequality corresponding to $1 \leq i_1 < \cdots < i_{k-1} < N-1 < i_k = N$ using the same indices $i_1, \ldots, i_k$, the fact that that $(d_1, \ldots, d_{N+1})$ is $D(G\triangle e)$-draconian and $\deg_G(N-1) = 2$ means that the left side of

$$d_{i_1} + \cdots + d_{i_{k-1}} + d_{N-1} + d_{i_k} < \left| \mathcal{N}_{D(G\triangle e)}(N-1) \cup \bigcup_{j=1}^{k} \mathcal{N}_{D(G\triangle e)}(i_j) \right|. \tag{9}$$

adds a value at least 1 while the right side of (9) is greater than that of (8) by at most 1. Since we know (9) holds, (8) must also hold. Therefore,

$$d_{i_1} + \cdots + d_{i_k} < \left| \bigcup_{j=1}^{k} \mathcal{N}_{D(G\triangle e)}(i_j) \right| - 1 = \left| \bigcup_{j=1}^{k} \mathcal{N}_{D(G)}(i_j) \right|.$$

This completes all cases for when $d_{N+1} = 0$.

Lastly, suppose $d_{N+1} = 2$. For this case we first show that $(d_1, \ldots, d_{N-1}+1, d_N+1) \in \mathcal{B}_G^{\triangle}(e)$. This can be rephrased as wanting to show $(d_1, \ldots, d_{N-1}+1, d_N+1) = \beta^{\triangle}(c)$ for some c, or, in yet other words, that $(d_1, \ldots, d_{N-1}, d_N+1) \in \mathfrak{D}(G)$.

This time set $d' = (d'_1, \ldots, d'_N) = (d_1, \ldots, d_{N-1}, d_N+1)$ and consider $1 \leq i_1 < \cdots < i_k \leq N$. If $i_k < N-1$ then the $D(G)$-draconian inequality holds as usual. If $i_k = N-1$ then observe

$$\begin{aligned} d'_{i_1} + \cdots + d'_{i_k} &< d_{i_1} + \cdots + d_{i_k} + 2 - 1 \\ &< \left| \left(\bigcup_{j=1}^{k} \mathcal{N}_{D(G)}(i_j) \right) \cup \mathcal{N}_{D(G)}(N) \right| - 1 \\ &= \left| \bigcup_{j=1}^{k} \mathcal{N}_{D(G)}(i_j) \right|. \end{aligned}$$

If $i_k = N$ then repeat this argument but by inserting "$+1-1$" instead of "$+2-1$". In all cases, the $D(G)$-draconian inequality holds, so $d' \in \mathfrak{D}(G)$, as needed. In fact, showing that $(d_1, \ldots, d_{N-2}, d_{N-1}+1, d_N)$ is $D(G)$-draconian has an entirely analogous argument. Therefore, this completes the case for $d_{N+1} = 2$.

By Lemma 3.17,

$$\mathfrak{D}(G\triangle e) = \mathcal{A}^{\triangle}_G(e) \uplus \mathcal{B}^{\triangle}_G(e) \uplus \mathcal{C}^{\triangle}_G(e).$$

Thus,

$$|\mathfrak{D}(G\triangle e)| = |\mathcal{A}^{\triangle}_G(e)| + |\mathcal{B}^{\triangle}_G(e)| + |\mathcal{C}^{\triangle}_G(e)| = 3|\mathfrak{D}(G)|.$$

Finally, by Theorem 2.8, we obtain

$$\mathrm{NVol}(\nabla^{\mathrm{PQ}}_{G\triangle e}) = 3\,\mathrm{NVol}(\nabla^{\mathrm{PQ}}_G).$$

□

Example 3.19. Let C_3 be the 3-cycle as in Example 3.11 and again choose $e = 13$. The $D(G\triangle e)$-draconian sequences are formed from the disjoint union of the three sets

$$\begin{aligned}
\mathcal{A}^{\triangle}_{C_3}(e) &= \{(2,0,0,1),(0,2,0,1),(0,0,2,1),(1,1,0,1),(1,0,1,1),(0,1,1,1)\}\\
\mathcal{B}^{\triangle}_{C_3}(e) &= \{(3,0,0,0),(1,2,0,0),(1,0,2,0),(2,1,0,0),(2,0,1,0),(1,1,1,0)\}\\
\mathcal{C}^{\triangle}_{C_3}(e) &= \{(1,0,0,2),(0,2,1,0),(0,0,3,0),(0,1,0,2),(0,0,1,2),(0,1,2,0)\}
\end{aligned}$$

As in the case of the subdivision recurrence, by relaxing the requirement that e has an endpoint of degree 2 in G, the result may no longer hold. The same example as before, where $G = K_1 \vee P_3$ and e is the edge whose endpoints each have degree 3 in G, demonstrates this. The normalized volume of $\nabla^{\mathrm{PQ}}_{G\triangle e}$ is 52 whereas a naive attempt to apply the triangle recurrence would predict 54.

Question 3.20. Under what conditions for a graph G and an edge e is there a “nice” recurrence for $\mathrm{NVol}(\nabla^{\mathrm{PQ}}_{G\triangle e})$?

3.3. **Application: Outerplanar graphs.** Recall that a *plane graph* is a planar graph G together with a particular embedding of G into the plane. Also recall that the *weak dual* of a plane graph G, denoted $G^{(*)}$, is the subgraph of the dual G^* induced by the vertices corresponding to bounded faces of G. We denote by E_k the *empty graph* on k vertices, that is, the disjoint union of k distinct vertices. Further, given a bounded face F, let $o_G(F)$ denote the number of edges of G bounding both F and the outer face and let v_F denote the vertex of $G^{(*)}$ corresponding to F. Let $\mathcal{F}(G)$ be the set of bounded faces of G.

Definition 3.21. Let G be a plane graph. The *extended weak dual* of G, denoted $G^{(**)}$, is

$$G^{(**)} = G^{(*)} \cup \left(\bigcup_{F\in\mathcal{F}(G)} v_F \vee E_{o(F)} \right)$$

Informally, $G^{(**)}$ extends the weak dual of G by including an additional edge for each edge of G that bounds the outer face. See Figure 2 for illustrations of a plane graph G and its duals $G^{(*)}, G^{(**)}$.

Recall that a graph is *outerplanar* if it has a planar embedding such that every vertex is incident to the outer face. It is known [7] that a graph is outerplanar if and only if its weak dual is a forest. Putting together the results of Section 3 we can produce a simple formula for $\mathrm{NVol}(\nabla^{\mathrm{PQ}}_G)$ whenever G is outerplanar and can be constructed inductively by using the subdivision and triangle operations. The formula follows quickly from the following theorem.

Theorem 3.22. Suppose G is a 2-connected plane graph on $[N]$ for which $G^{(**)}$ is a tree and every vertex of $G^{(**)}$ is a leaf or adjacent to at least one leaf. Then

$$\mathrm{NVol}(\nabla^{\mathrm{PQ}}_G) = 2^{N-2|\mathcal{F}(G)|} \prod_{F\in\mathcal{F}(G)} \deg_{G^{(**)}}(v_F).$$

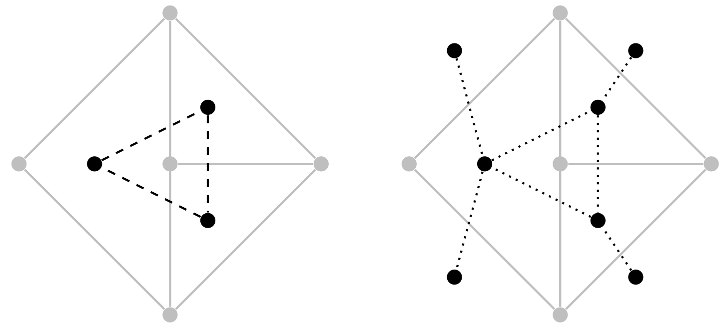

FIGURE 2. A graph G (gray) with its weak dual $G^{(*)}$ superimposed (left, dashed) and with its extended weak dual $G^{(**)}$ superimposed (right, dotted).

Proof. We induct on the number of internal vertices of $G^{(**)}$, that is, $|V(G^{(*)})|$. If $|V(G^{(*)})| = 1$, then $G = C_N$ for some N. By Corollary 3.12,

$$\mathrm{NVol}(\nabla^{\mathrm{PQ}}_{C_N}) = N2^{N-2}.$$

Since $|\mathcal{L}({C_N}^{(**)})| = N$ and $|\mathcal{F}(C_N)| = 1$, the formula holds.

Now suppose there are $k > 1$ internal vertices of $G^{(**)}$. Choose a vertex w of $G^{(**)}$ which is adjacent to a leaf, and let S be the leaves of $G^{(**)}$ adjacent to w, that is,

$$S = \mathcal{N}_{G^{(**)}}(w) \cap \mathcal{L}(G^{(**)}).$$

Let G' be the subgraph of G whose extended weak dual is $G^{(**)} - S$. Here, w is a leaf, so $G^{(**)} - S$ is still a tree where each vertex is a leaf or adjacent to a leaf. By the inductive assumption,

$$\begin{aligned}\mathrm{NVol}(\nabla^{\mathrm{PQ}}_{G'}) &= 2^{|\mathcal{L}({G'}^{(**)})|-2|\mathcal{F}(G')|} \prod_{F\in\mathcal{F}(G')} \deg_{G'^{(**)}}(v_F)\\ &= 2^{|\mathcal{L}(G^{(**)})|-|S|-2|\mathcal{F}(G)|+2} \prod_{F\in\mathcal{F}(G')} \deg_{G'^{(**)}}(v_F).\end{aligned}$$

To obtain the desired formula, we must show that by applying appropriate recurrences, we end up multiplying the above by

$$2^{|S|-2} \deg_{G^{(**)}}(w).$$

Doing so is a straightforward application of the triangle recurrence and $|S| - 2$ applications of the subdivision recurrence. □

Theorem 3.22 is the final piece needed to compute $\mathrm{NVol}(\nabla^{\mathrm{PQ}}_G)$ for any outerplane graph whose 2-connected components satisfy the conditions of Theorem 3.22.

Corollary 3.23. Let G be any outerplane graph on $[N]$ such that each bounded face is adjacent to the outer face. Label its components $G_1, \ldots, G_k$ and let $B_{i,1}, \ldots, B_{i,b_i}$ be the blocks of G_i. Then

$$\mathrm{NVol}(\nabla^{\mathrm{PQ}}_G) = \prod_{i=1}^{k}\prod_{j=1}^{b_i} 2^{|V(B_{i,j})|-2|\mathcal{F}(B_{i,j})|} \prod_{F\in\mathcal{F}(B_{i,j})} \deg_{{B_{i,j}}^{(**)}}(v_F). \tag{10}$$ □

The graphs satisfying the conditions needed in Corollary 3.23 form a proper, but large, class of outerplane graphs. Experimental data suggests that the formula is, in fact, true for all outerplane graphs, but a proof eludes the authors.

Conjecture 3.24. For *any* outerplane graph G, Equation (10) holds.

4. Beyond outerplanarity

Outerplanar graphs form a large class of graphs but are far from the class of planar graphs, let alone all graphs. For example, even though there are about 56.7×10^9 labeled outerplanar graphs on 10 vertices, these account for only approximately 1.76% of all labeled planar graphs on 10 vertices [8, Sequences A098000, A066537]. Because of the difficulty in computing $\mathrm{NVol}(\nabla_G^{\mathrm{PQ}})$ for all graphs, a natural next step would be to consider graphs that are not-quite-outerplanar. Toward this end, we use the following alternate characterization of outerplanar graphs.

Theorem 4.1 ([1, Theorem 10.24])**.** A graph is outerplanar if and only if contains no subdivision of K_4 or $K_{2,3}$ as a subgraph.

This is a direct analogue of Kuratowski's theorem, allowing one to study graphs G that contain no subdivision of K_5 or $K_{3,3}$ but may contain a subdivision of K_4 or $K_{2,3}$. In this case, a formula for $|\mathfrak{D}(G)|$ remains elusive, although we do have the following partial result. We use the notation $K^0_{M,N}$ to denote the complete bipartite graph with partite sets $[0, \dots, M-1]$ and $[M, M+N-1]$.

Proposition 4.2. For all $N \geq 3$,
$$\mathrm{NVol}(\nabla^{\mathrm{PQ}}_{K_{2,N-2}}) = 2^{N-4}(N^2 - N + 6) - 2.$$

Proof. If $(c_1, \dots, c_N) \in \mathfrak{D}(K_{2,N-2})$ then $c_1 + c_2 = k$ for some $0 \leq k \leq N-1$. All possible choices of c_1, c_2 are part of a $D(K_{2,N-2})$-draconian sequence except for $(c_1, c_2) \in \{(N-1, 0), (0, N-1)\}$ since these are the only two resulting in sequences not satisfying the corresponding draconian inequalities. However, for the moment, we will include these in our calculations for algebraic ease.

In order to satisfy the $D(K_{2,N-2})$-draconian inequalities we need the subsequence $c' = (c_3, \dots, c_N)$ to be a weak composition of $N-1-k$ using 0s, 1s, and 2s such that there is at most one 2. This leads to two cases: if c' contains a 2, then there must be $N-3-k$ copies of 1 and k copies of 0. A simple counting argument gives
$$(N-2)\binom{N-3}{k}$$
such possibilities. On the other hand if c' does not contain any 2s, then there must be $N-1-k$ copies of 1 and $k-1$ copies of 0. There are $\binom{N-2}{k-1}$ such possibilities. Adding the values from these two cases and summing over all k yields
$$\sum_{k=0}^{N-1}(k+1)\left((N-2)\binom{N-3}{k} + \binom{N-2}{k-1}\right).$$
The reader may verify that this simplifies to $2^{N-4}(N^2-N+6)$. Subtracting the two compositions where $(c_1, c_2) \in \{(N-1, 0), (0, N-1)\}$ and applying Theorem 2.8 gives us our final formula. $\square$

Question 4.3. What is $\mathrm{NVol}(\nabla^{\mathrm{PQ}}_{K_{M,N}})$ for arbitrary M, N?

Notice that the formula in Proposition 4.2 cannot be written in the form of (10). Thus, a general formula for planar graphs will require refining the techniques of Section 3 or separate tools altogether.

A second important class of graphs which are planar but not outerplanar is the class of *wheel graphs* $W_N = K_1 \vee C_N$. We conjecture the following.

Conjecture 4.4. For all $N \geq 3$,

$$\mathrm{NVol}(\nabla^{\mathrm{PQ}}_{W_N}) = 3^N - 2^N + 1.$$

This conjecture has been verified for all $3 \leq N \leq 13$. Wheels were examined in detail in [5] within a related, but distinct, context from $\nabla^{\mathrm{PQ}}_{W_N}$. We hope to uncover similarly rich structure in the present setting. It may be useful to recognize that

$$3^N - 2^N + 1 = 2S(N+1, 3) + S(N+1, 2) + S(N+1, 1),$$

where $S(n, k)$ denotes the Stirling number of the second kind.

Finally, we give another broad class of graphs which contains all outerplanar graphs but not all planar graphs. Strictly speaking, these graphs will allow for repeated edges, but as repeating an edge in G does not affect ∇^{PQ}_G, we need not worry about that case.

Following [6], first consider the directed graphs formed in the following way. Begin with a single edge and designate one vertex the source and another vertex the sink. This is an example of a *two-terminal series-parallel graph*. All other two-terminal series-parallel graphs are those formed by applying one of the following operations to two existing two-terminal series-parallel graphs G and H with sources g and h and sinks g' and h', respectively,

1. *parallel composition*: produce a new graph $\mathcal{P}(G, H)$ by identifying g with h and g' with h'. The source of $\mathcal{P}(G, H)$ is $g \sim h$ and its sink is $g' \sim h'$.
2. *series composition*: produce a new graph $\mathcal{S}(G, H)$ by identifying g' with h. The source of $\mathcal{S}(G, H)$ is g and its sink is h'.

A graph G is a *series-parallel graph* if there are two vertices x, y such that, when designating x as the source and y as the sink, G can be obtained through a sequence of applications of $\mathcal{P}(\cdot, \cdot)$ and $\mathcal{S}(\cdot, \cdot)$ when starting with a disjoint union of edges.

Series-parallel graphs are of interest in computer algorithms, as recognizing them is difficult but not intractable. For our purposes, they are of interest because their recursive structure suggests that they may be good candidates for computing $\mathrm{NVol}(\nabla^{\mathrm{PQ}}_G)$. In fact, we have already seen an example of a series-parallel graph: $K_{2,N-2}$ is the parallel composition of $N-2$ copies of P_3, each of which is a series composition of two edges. We ask the following question broadly, and would be interested in seeing answers to even nontrivial subclasses which are not outerplanar.

Question 4.5. What is $\mathrm{NVol}(\nabla^{\mathrm{PQ}}_G)$ for a series-parallel graph G?

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