On the equality of BKK bound and birationally invariant intersection index*

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Abstract. The Bernshtein-Kushnirenko-Khovanskii theorem provides a generic root count for system of Laurent polynomials in terms of the mixed volume of their Newton polytopes which has since been known as the BKK bound. A recent and far-reaching generalization of this theorem is the study of birationally invariant intersection index by Kaveh and Khovanskii. In this paper we generalize the BKK bound in the direction of the birationally invariant intersection index, and the main result allows the application of BKK bound to Laurent polynomial systems that has algebraic relations among the coefficients. Applying this result, we establish the birationally invariant intersection index for a well-studied algebraic Kuramoto equations.

Key words. BKK bound, Bernshtein-Kushnirenko-Khovanskii theorem, birationally invariant intersection index, Newton polytope, mixed volume, Kuramoto equations

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1. Introduction.

The famous Bernshtein-Kushinirenko-Khovanskii theorem [2, 9, 10, 12, 13] relates the root counting problem for system of polynomial equations and the theory of convex bodies. In particular, it shows that the generic number of isolated solutions a system of Laurent polynomial equations has in the algebraic torus $(\mathbb{C}^*)^n = (\mathbb{C} \setminus \{0\})^n$ equals the mixed volume of the Newton polytopes of the Laurent polynomials.

Recently, a far-reaching generalization of this theorem is developed in a series of works [6, 7, 8] by K. Kaveh and A. Khovanskii where the root counting question is considered for much more general spaces of rational functions. Given an irreducible n-dimensional complex algebraic variety X and n-tuple of finite dimensional vector spaces $(L_1, \ldots, L_n)$ of rational functions on X, for generic elements $f_i \in L_i$ for $i = 1, \ldots, n$, the number of common solutions a system $f_1 = \cdots = f_n = 0$ has in X is a constant, and it is given by the mixed volume of Newton-Okunkov boides associated with $L_1, \ldots, L_n$. This generic root count is given the name *birationally invariant intersection index*.

In this paper we generalized the theory of Bernshtein-Kushnirenko-Khovanskii bound (or BKK bound) toward the direction of the birationally invariant intersection index: we show that under certain conditions, the birationally invariant intersection index coincide with the BKK bound even though the space of functions are not generated by monomials, but instead are spans of Laurent polynomials.

This paper is structured as follows. Section 2 review necessary concepts and notations. In section 3 we establish the main theorem. An application of this theorem to the well-studied Algebraic Kuramoto equations is described in section 4, and we conclude in section 5.

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2. Preliminaries. A *Laurent monomial* in $\mathbf{x} = (x_1, \ldots, x_n)$ is a product $\mathbf{x}^{\mathbf{a}} = x_1^{a_1} \cdots x_n^{a_n}$ for some $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{Z}^n$. A *Laurent polynomial* is a $\mathbb{C}$- linear combination of Laurent monomials of the form $f(\mathbf{x}) = \sum_{\mathbf{a} \in S} c_{\mathbf{a}} \mathbf{x}^{\mathbf{a}}$ where the finite set $S \subset \mathbb{Z}^n$, collecting all the exponent vectors, is its *support*. The convex hull $\operatorname{conv}(S)$ of the support is the *Newton polytope* of f, denoted by $\operatorname{Newt}(f)$. With respect a vector $\mathbf{v} \in \mathbb{R}^n$, its *initial form* $\operatorname{init}_{\mathbf{v}}(f)$ is the expression $\sum_{\mathbf{a} \in (S)_{\mathbf{v}}} c_{\mathbf{a}} \mathbf{x}^{\mathbf{a}}$ where $(S)_{\mathbf{v}} \subset S$ is the subset on which the linear functional $\langle \mathbf{v}\,,\,\cdot \rangle$ is minimized over S. $\mathbb{C}[x_1^{\pm}, \ldots, x_n^{\pm}]$ denotes the set of all Laurent polynomials in $x_1, \ldots, x_n$.

Computing the number of common roots for a system of Laurent polynomials (a Laurent polynomial system) is an important problem in algebraic geometry. Since the exponents may be negative, the natural space to study this root counting question is the *algebraic torus* $(\mathbb{C}^*)^n = (\mathbb{C} \setminus \{0\})^n$. While the root count of a Laurent polynomial system can vary greatly depending on the coefficients, for "generic" coefficients, however, the $(\mathbb{C}^*)^n$-root count remains a constant and only depends on the monomial structure of the system. D. Bernshtein showed this constant is precisely the "mixed volume" of the Newton polytopes. Given two sets $A, B \subset \mathbb{R}^n$, their *Minkowski sum* is $A + B = \{\mathbf{a} + \mathbf{b} \mid \mathbf{a} \in A, \mathbf{b} \in B\}$. For convex polytopes $P_1, \ldots, P_n \subset \mathbb{R}^n$, the volume $\operatorname{Vol}(\lambda_1 P_1 + \cdots + \lambda_n P_n)$ is a homogeneous polynomial in $\lambda_1, \ldots, \lambda_n$ [14]. The coefficient of the mixed term $\lambda_1 \cdots \lambda_n$ is known as the *mixed volume* of these polytopes, denoted by $\operatorname{MVol}(P_1, \ldots, P_n)$.

Theorem 1 (D. Bernshtein 1975 [2]). *Given a Laurent polynomial system* $\mathbf{f} = (f_1, \ldots, f_n)$ *with* $f_i(\mathbf{x}) = \sum_{\mathbf{a} \in S_i} c_{i,\mathbf{a}} \mathbf{x}^{\mathbf{a}}$ *where* $\mathbf{x} = (x_1, \ldots, x_n)$*, if for all nonzero vectors* $\mathbf{v} \in \mathbb{R}^n$*, the initial system* $\operatorname{init}_{\mathbf{v}}(\mathbf{f})$ *has no zero in* $(\mathbb{C}^*)^n$*, then all zeros of* $\mathbf{f}$ *in* $(\mathbb{C}^*)^n$ *are isolated, and the total number, counting multiplicity, is the mixed volume* $\operatorname{MVol}(\operatorname{conv}(S_1), \ldots, \operatorname{conv}(S_n))$.

Lemma 2 (D. Bernshtein 1975 [2]). *Given a Laurent polynomial system* $\mathbf{f} = (f_1, \ldots, f_n)$ *with* $f_i(\mathbf{x}) = \sum_{\mathbf{a} \in S_i} c_{i,\mathbf{a}} \mathbf{x}^{\mathbf{a}}$ *where* $\mathbf{x} = (x_1, \ldots, x_n)$*, for generic choices of the coefficients* $\{c_{i,\mathbf{a}}\}$*, the initial system* $\operatorname{init}_{\mathbf{v}}(\mathbf{f})$ *has no solution in* $(\mathbb{C}^*)^n$ *for any nonzero vector* $\mathbf{v} \in \mathbb{R}^n$.

This bound is known as the *Bernshtein-Kushnirenko-Khovanskii* (BKK) bound, after a circle of closely related works by Bernshtein [2], Kushnirenko [10, 12, 13], and Khovanskii [9]. Here, the notion of "generic choices" is to be interpreted in terms of *Zariski topology* — within the space of all possible coefficients, there is a Zariski-open set for which this bound is exact.

Recently, this result is generalized considerably into the theory of *birationally invariant intersection index*. Instead of considering Laurent polynomials with generic coefficients, which can be thought of as generic linear combinations of Laurent monomials, one could consider generic linear combinations of rational functions with more structure. In the most general setting, as studied in [7, 8], one starts with $\mathbb{C}$-vector spaces $L_1, \ldots, L_n$ where each L_i is the span of finitely many rational functions $Q_{i,1}, \ldots, Q_{i,m_i}$ on an irreducible toric variety X, then for generic choices of functions $f_1 \in L_1, \ldots, f_n \in L_n$, the number of common isolated solutions of $(f_1, \ldots, f_n) = \mathbf{0}$ in X is a constant that is independent of the choices. That number, is known as the *birationally invariant intersection index* of $L_1, \ldots, L_n$ and is denoted by $[L_1, \ldots, L_n]$. This grand theory relates the root counting problem to the geometric properties of *Newton-Okunkov bodies*, and the BKK bound is thus a special case of this intersection index in the situations where each L_i is spanned by Laurent monomials. In the following, we extend the BKK bound to certain cases where each L_i is spanned by Laurent polynomials.

3. The main theorem. The goal here is to show the equality of birationally invariant intersection index and the BKK bound under certain geometric conditions and thereby generalize the theory of BKK bound. We focus on the cases where $X = (\mathbb{C}^*)^n$ and $L_1, \dots, L_n$ are vector spaces of rational functions spanned by finitely many Laurent polynomials. That is,

$$L_i = \operatorname{span}_{\mathbb{C}}\{P_{ij}\}_{j=1}^{m_i} \tag{1}$$

where each $P_{ij} \in \mathbb{C}[x_1^{\pm}, \dots, x_n^{\pm}]$ and $m_i \in \mathbb{Z}^+$. This setup is a generalization of the situation in Theorem 1 where each L_i is only spanned by a set of Laurent monomials. That is, if each P_{ij} is a Laurent monomial, then $[L_1, \dots, L_n]$ is exactly the BKK bound. The main result here is a generalization the BKK bound to include cases where each P_{ij} is a Laurent polynomial.

A generic element $f_i \in L_i$, is a Laurent polynomial $f_i = \sum_{j=1}^{m_i} c_{ij} P_{ij}$ with generic choice of the coefficients $c_{i1}, \dots, c_{im_i}$. It is easy to see that among the terms within such a generic element, there is no cancellations and consequently $\operatorname{Newt}(f_i) = \operatorname{conv}\left(\cup_{j=1}^{m_i} \operatorname{Newt}(P_{ij})\right)$. It is therefore reasonable to define

$$\operatorname{Newt}(L_i) = \operatorname{conv}\left(\cup_{j=1}^{m_i} \operatorname{Newt}(P_{ij})\right).$$

Theorem 3. *Let $L_1, \dots, L_n$ be vector spaces of rational functions with $L_i = \operatorname{span}_{\mathbb{C}}\{P_{ij}\}_{j=1}^{m_i}$ where each $P_{ij} \in \mathbb{C}[x_1^{\pm}, \dots, x_n^{\pm}]$ and $m_i \in \mathbb{Z}^+$ as described above. If for each $i = 1, \dots, n$, $\dim(\operatorname{Newt}(L_i)) = n$ and every positive-dimensional proper faces of $\operatorname{Newt}(L_i)$ intersect $\operatorname{Newt}(P_{ij})$ at no more than one point for each $j = 1, \dots, m_i$, then*

$$[\, L_1, \dots, L_n \,] = \operatorname{MVol}(\operatorname{Newt}(L_1), \dots, \operatorname{Newt}(L_n)). \tag{2}$$

Proof. Let $f_1, \dots, f_n$ be generic elements in $L_1, \dots, L_n$ respectively, i.e., $f_i = \sum_{j=1}^{m_i} c_{ij} P_{ij}$ for generic coefficients $\{c_{ij}\}$. Then the common root count of the system $\mathbf{f} = (f_1, \dots, f_n)$ in $(\mathbb{C}^*)^n$ equals $[L_1, \dots, L_n]$. It is therefore sufficient to show the root count of $\mathbf{f}$ in $(\mathbb{C}^*)^n$ matches the BKK bound, i.e., $\mathbf{f}$ satisfies the conditions in Theorem 1.

Let $\mathbf{v} \in \mathbb{R}^n$ be a nonzero vector such that $\operatorname{init}_{\mathbf{v}}(\mathbf{f})$ does not contain a unit (i.e., no component of $\mathbf{f}$ is a single Laurent monomial term). Since $\operatorname{Newt}(f_i) = \operatorname{Newt}(L_i)$ is assumed to be full-dimensional for $i = 1, \dots, n$, $\mathbf{v}$ must be a common inner normal vector for n proper positive dimensional faces $F_1, \dots, F_n$ of $\operatorname{Newt}(f_1), \dots, \operatorname{Newt}(f_n)$ respectively.

For each $i = 1, \dots, n$, let $A_{ij} = F_i \cap \operatorname{Newt}(P_{ij})$, then, by assumption, each A_{ij} contains at most one point. Without loss of generality, after re-indexing P_{ij}'s, we can assume that for a fixed i, $A_{ij} = \{\mathbf{a}_{ij}\}$ for $j = 1, \dots, m_i'$ and $A_{ij} = \varnothing$ for $j = m_i' + 1, \dots, m_i$ where $m_i' \in \mathbb{Z}^+$ and $m_i' \le m_i$ (since $F_i \cap \operatorname{Newt}(P_{ij})$ may be empty for some j). With this definition, $\{\mathbf{a}_{i,1}, \dots, \mathbf{a}_{i,m_i'}\} = \bigcup_{j=1}^{m_i} A_{ij}$, and consequently,

$$\operatorname{init}_{\mathbf{v}}(f_i) \in \operatorname{span}_{\mathbb{C}}\{\mathbf{x}^{\mathbf{a}_{i1}}, \dots, \mathbf{x}^{\mathbf{a}_{im_i'}}\}.$$

More importantly, we can see the set of coefficients is a subset of the coefficients in f_i. Indeed,

$$\operatorname{init}_{\mathbf{v}}(f_i) = \sum_{j=1}^{m_i'} c_{ij} \mathbf{x}^{\mathbf{a}_{i,j}}$$

with the exponent vectors $\mathbf{a}_{i1}, \ldots, \mathbf{a}_{im'_i}$ all lie in a proper face of $\mathrm{Newt}(L_i)$ and the coefficients c_{ij}'s being independent from one another. By Lemma 2, there exists a nonempty Zariski open set in the coefficient space $\{c_{ij}\}$ for which the initial system $\mathrm{init}_{\mathbf{v}}(\mathbf{f})$ has no solution in $(\mathbb{C}^*)^n$.

Note that there are only finitely many distinct initial systems for $\mathbf{f}$. By taking the intersection of a finite number of nonempty Zariski open set, we can see that there remains a nonempty Zariski open set in the coefficient space $\{c_{ij}\}$ such that for all choices in this set, $\mathrm{init}_{\mathbf{v}}(\mathbf{f})$ either contains a unit or has no solution in $(\mathbb{C}^*)^n$ for any nonzero vector $\mathbf{v} \in \mathbb{R}^n$.

By Theorem 1, for generic choices of the coefficients $\{c_{ij}\}$, the BKK bound for $\mathbf{f}$ is exact, i.e., the common root count in $(\mathbb{C}^*)^n$ for this system is exactly $\mathrm{MVol}(\mathrm{Newt}(f_1), \ldots, \mathrm{Newt}(f_n))$. Recall that each f_i is a generic member of L_i. This shows

$$[\, L_1, \ldots, L_n \,] = \mathrm{MVol}(\, \mathrm{Newt}(L_1), \ldots, \mathrm{Newt}(L_n)\,).$$ ■

4. Application to Kuramoto equations. The Kuramoto model [11] is a ubiquitous model for studying the phenomenon of spontaneous synchronization of a network of coupled oscillators, and it has found important applications in many independent fields of studies. The algebraic synchronization equation [1, 3, 5, 4] for a Kuramoto model of $n+1$ oscillators is a system of n Laurent polynomial equations in the n variables $\mathbf{x} = (x_1, \ldots, x_n)$ given by

$$f_i(\mathbf{x}) = \omega_i - \sum_{j=0}^{n} a_{ij} \left(\frac{x_i}{x_j} - \frac{x_j}{x_i} \right) \quad \text{for } i = 1, \ldots, n \tag{3}$$

where $\{\omega_i\}_{i=1}^n$ and $\{a_{ij}\}_{i,j \in \{0,\ldots,n\}}$ are complex constants, and $x_0 = 1$.

Theorem 4. *For generic choices of the complex constants $\{\omega_i\}_{i=1}^n$ and $\{a_{ij}\}_{i,j\in\{0,\ldots,n\}}$, the number of isolated complex solutions to the system (3) is exactly the BKK bound of the system.*

Proof. For each $i \in \{1, \ldots, n\}$ and $j \in \{0, \ldots, n\}$ we define

$$P_{ij} = \frac{x_i}{x_j} - \frac{x_j}{x_i}.$$

Then each P_{ij} is a Laurent polynomial in the variables $\mathbf{x} = (x_1, \ldots, x_n)$, and $\mathrm{Newt}(P_{ij}) = \mathrm{conv}\{\mathbf{e}_i - \mathbf{e}_j, \mathbf{e}_j - \mathbf{e}_i\}$ where $\mathbf{e}_0 = \mathbf{0}$. So for each pair (i, j), $\mathrm{Newt}(P_{ij})$ is a line segment through the origin.

Now consider the vector space of rational functions

$$L_i = \mathrm{span}_{\mathbb{C}}\ \{1\} \cup \{P_{ij}\}_{j=0}^n.$$

It is easy to verify that $f_i \in L_i$ for each $i = 1, \ldots, n$. Therefore the statement to be proved is equivalent to the claim that $[L_1, \ldots, L_n]$ equals the BKK bound of the system $(f_1, \ldots, f_n)$.

By definition,

$$\mathrm{Newt}(L_i) = \mathrm{conv}(\mathbf{0}\ \cup\ \{\mathbf{e}_i - \mathbf{e}_j\,,\, \mathbf{e}_j - \mathbf{e}_i\}_{j=0}^n),$$

and, in it, $\mathbf{0}$ is an interior point. For $n > 1$, $\mathrm{Newt}(L_i)$ is the convex hull of n affinely independent line segments through the origin, and thus $\dim(\mathrm{Newt}(L_i)) = n$ for every i. Moreover, fixing i, for each $j = 0, \ldots, n$ and $j \neq i$, $\mathrm{Newt}(P_{ij})$ is a line segment passing though an interior point, the origin, of $\mathrm{Newt}(L_i)$. Therefore for each proper positive dimensional face F of $\mathrm{Newt}(L_i)$, $F \cap \mathrm{Newt}(P_{ij})$ is either empty or a single point. By Theorem 3, the generic root count in $(\mathbb{C}^*)^n$, i.e., $[L_1, \ldots, L_n]$ is exactly the BKK bound. ■

5. Conclusions. In this short paper, we extend the classical BKK bound theory to certain cases where each Laurent polynomial is a generic linear combination of several Laurent polynomials. The main result (Theorem 3) establishes a sufficient condition under which the birationally invariant intersection index equals exactly the BKK bound. This condition is stated purely in terms of the combinatorial information in the Newton polytopes of the polynomials involved and can be checked easily using simple algorithms from convex geometry. It shows that certain algebraic relations among the coefficients have no effect on the exactness of the BKK bound. The usefulness of this result is demonstrated through an application to the algebraic Kuramoto equations — a well studied family of equations used to model spontaneous synchronization phenomenon in many fields. With this theorem, we easily established a previously unknown fact: the BKK bound agrees with the generic number of complex solutions even though this system has inherent algebraic relations among the coefficients.

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